Fluid moments in modified guiding-centre coordinates

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A general fluid moment in the particle coordinates is represented in terms of fluid moments in modified guiding-centre coordinates with strong $\mathbf{E}\times\mathbf{B}$ flow derived by the phase space Lagrangian Lie-transform method [N. Miyato et al., J. Phys. Soc. Jpn. **78**, 104501 (2009)]. It is called the push-forward representation of the particle fluid moment associated with the guiding-centre transformation. The representation derived is similar to that in terms of gyro-fluid moments in the standard gyrokinetic theory in the long wavelength limit. Since the particle coordinates are transformed to gyro-centre coordinates by the two-step transformation in the standard gyrokinetic formulation, two exact push-forward representations are possible. Although the exact representation usually used in the standard gyrokinetic theory has a different form from that in the modified guiding-centre case, the correspondence between the two cases is shown more clearly by considering the alternative form of the push-forward representation for the standard gyrokinetic case.

Keywords: phase space transformation, push-forward representation, flow

1. Introduction

It is well known that guiding-centre or gyro-centre density is different from particle density due to finite-Larmor-radius (FLR) effects [1–8]. Generally guiding-centre or gyro-centre fluid moments are different from corresponding particle fluid moments. Any particle fluid moment can be represented in terms of the guiding-centre or gyro-centre fluid moments. It is called the push-forward representation of the particle fluid moment associated with the transformation from the particle phase space to the guiding-centre or gyro-centre phase space [5, 6]. The inverse of the representation was used to derive nonlinear reduced fluid equations with FLR corrections from nonlinear gyrofluid equations [1]. Since the gyromotion of a charged particle is removed at the kinetic level, this procedure allows one to bypass the issue of gyroviscous cancellations and corrections in traditional derivations where an explicit representation of the stress tensor is needed [9, 10]. An expression for the gyroviscous force is obtained by comparing the reduced equations with the FLR corrections obtained from the gyrofluid equations with the particle fluid momentum equation [3]. The correspondence between the gyrofluid and low-frequency fluid equations is also shown by using the relation between the gyrofluid moments and the particle fluid moments [4]. Recently we derived a modified guiding-centre fundamental 1-form with strong $\mathbf{E}\times\mathbf{B}$ flow from which a guiding-centre Vlasov-Poisson system was also constructed through the field theory [11]. It is of interest to investigate a relation between fluid moments in the modified guiding-centre coordinates and the particle-fluid moments. In contrast to conventional formulations with strong $\mathbf{E}\times\mathbf{B}$ flow [12–16], the symplectic part of the guiding-centre 1-form in our formulation does not include the $\mathbf{E}\times\mathbf{B}$ drift velocity term and is the same as that in the standard gyrokinetic model formally [5]. The guiding-centre Hamiltonian also agrees with the standard gyrokinetic Hamiltonian in the long wavelength limit. Therefore it is expected that the relation between the fluid moments in the modified guiding-centre coordinates and the particle fluid moments be similar to the one obtained from the standard gyrokinetic model in the long wavelength limit. In this paper, we represent the particle fluid moments in terms of the fluid moments in the modified guiding-centre coordinates. The representation is compared with that obtained from the standard gyrokinetic model in the long wavelength limit.

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2. Guiding-centre theory

We consider a transformation from particle coordinates $(\mathbf{x}, u, w, \theta)$ to guiding-centre coordinates $(\mathbf{X}, U, \mu, \xi)$ given by [7, 11]

$$\mathbf{X} = \mathbf{x} - \epsilon\boldsymbol{\rho} - \epsilon\boldsymbol{\rho}_E + O(\epsilon^2), \quad (1)$$

$$U = v_{\parallel} + O(\epsilon), \quad (2)$$

$$\mu = \frac{mw^2}{2B_0} + O(\epsilon), \quad (3)$$

$$\xi = \theta + O(\epsilon), \quad (4)$$

where $\mathbf{x}$ is the position of a particle with mass m and electric charge q, $v_{\parallel}$ is the parallel velocity, w is the perpendicular velocity in the frame moving with $\mathbf{D}$ which is the $\mathbf{E}\times\mathbf{B}$ drift velocity, θ is the gyrophase, $\boldsymbol{\rho}$ is the Larmor radius evaluated by w, $\boldsymbol{\rho}_E = \hat{b}\times\mathbf{D}/\Omega$, $\Omega = qB_0/m$ and $\epsilon \sim \rho/L$ is the small parameter with the background gradient scale length L. Here it is assumed that the $\mathbf{E}\times\mathbf{B}$ drift velocity is comparable to the ion thermal velocity v_{ti}. Therefore, $\boldsymbol{\rho}_E \sim \boldsymbol{\rho}$ for ions. The guiding-centre transformation for $\mathbf{X}$ is different from conventional ones in which the guiding centre position $\mathbf{X}'$ is defined by $\mathbf{X}' \equiv \mathbf{x} - \epsilon\boldsymbol{\rho}$ [12–16]. $\mathbf{X}$ is mainly shifted by $\boldsymbol{\rho}_E$ from $\mathbf{X}'$. The guiding-centre fundamental 1-form is given by

$$\Gamma = \left[q\mathbf{A}_0 + \epsilon mU\hat{b} - \epsilon^2\frac{m}{q}\mu\mathbf{W}\right]\cdot d\mathbf{X} + \epsilon^2\frac{m}{q}\mu d\xi - Hdt, \quad (5)$$

where $\mathbf{W} = \mathbf{R} + (\hat{b}\cdot\nabla\times\hat{b})\hat{b}/2$, $\hat{b} \equiv \mathbf{B}_0/B_0$, $\mathbf{R} \equiv (\nabla\mathbf{e}_1)\cdot\mathbf{e}_2$, $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors spanning the plane perpendicular to $\hat{b}$, and the guiding-centre Hamiltonian is

$$H = q\phi + \epsilon\left(\frac{m}{2}U^2 + \mu B_0 - \frac{m}{2}D^2\right) + \epsilon^2\left[\frac{m}{2q}\left(\mu + \frac{mD^2}{2B_0}\right)\hat{b}\cdot\nabla\times\mathbf{D}\right]. \quad (6)$$

The guiding-centre Hamilton equations, $\dot{Z}^i = \{Z^i, H\}$, are written as

$$\dot{\mathbf{X}} = \epsilon^{-1}\frac{\mathbf{B}^*}{mB^*_\parallel}\frac{\partial H}{\partial U} + \frac{\hat{b}}{qB^*_\parallel}\times\nabla H, \quad (7)$$

$$\dot{U} = -\epsilon^{-1}\frac{\mathbf{B}^*}{mB^*_\parallel}\cdot\nabla H, \quad (8)$$

$$\dot{\mu} \equiv 0, \quad (9)$$

$$\dot{\xi} = \epsilon^{-2}\frac{q}{m}\frac{\partial H}{\partial\mu} + \mathbf{W}\cdot\dot{\mathbf{X}}, \quad (10)$$

where $\mathbf{A}^* = \mathbf{A}_0 + (m/q)U\hat{b}$, $\mathbf{B}^* = \nabla\times\mathbf{A}^*$ and $B^*_\parallel = \hat{b}\cdot\mathbf{B}^*$. These are the same as those in the standard gyrokinetic model formally.

3. Push-forward representation of fluid moments

3.1 Standard gyrokinetic case

Following Belova [3], we brief the relation between the gyro-centre fluid moments and the particle fluid moments first. A general particle fluid moment is defined by

$$m_{kl}(\mathbf{r}) \equiv \int\left(\frac{mv_\perp^2}{2B_0}\right)^k v_\parallel^l f\delta^3(\mathbf{x}-\mathbf{r})d^3\mathbf{x}d^3\mathbf{v}. \quad (11)$$

This particle fluid moment can be written in terms of the gyro-centre distribution function $\overline{F}$ in the electrostatic limit as

$$\begin{aligned} m_{kl}(\mathbf{r}) &= \int d^6\overline{\mathbf{Z}}\mathcal{J}(\overline{\mathbf{Z}}) \\ &\quad\times\left[\mathsf{T}_{\rm GC}^{-1*}\left\{\left(\frac{mv_\perp^2}{2B_0}\right)^k v_\parallel^l\right\}\right](\overline{\mathbf{Z}}) \\ &\quad\times[\mathsf{T}_{\rm Gy}^*\overline{F}](\overline{\mathbf{Z}})\delta^3(\mathsf{T}_{\rm GC}^{-1}\overline{\mathbf{X}}-\mathbf{r}) \\ &\simeq \int d^6\overline{\mathbf{Z}}\mathcal{J}(\overline{\mathbf{Z}})\overline{\mu}^k\overline{U}^l\left[\overline{F} + \epsilon_\delta\frac{q\tilde{\phi}}{B_0}\frac{\partial\overline{F}}{\partial\overline{\mu}}\right] \\ &\quad\times\delta^3(\overline{\mathbf{X}}+\overline{\boldsymbol{\rho}}-\mathbf{r}) \end{aligned} \quad (12)$$

where $\overline{\mathbf{Z}} = (\overline{X}, \overline{U}, \overline{\mu}, \overline{\xi})$ are the gyro-centre coordinates, $d^6\overline{\mathbf{Z}} = d\overline{\mathbf{X}}d\overline{U}d\overline{\mu}d\overline{\xi}$, $\mathcal{J} = B^*_\parallel/m$ is the Jacobian, $\mathsf{T}_{\rm GC}^{-1}$ is the inverse of the transformation from the particle phase space to the guiding-centre phase space, $\mathsf{T}_{\rm GC}^{-1*}$ is the pull-back transformation associated with $\mathsf{T}_{\rm GC}^{-1}$ or the push-forward transformation associated with $\mathsf{T}_{\rm GC}$, $\mathsf{T}_{\rm Gy}$ is the transformation from the guiding-centre phase space to the gyro-centre phase space, $\mathsf{T}_{\rm Gy}^*$ is the pull-back transformation associated with $\mathsf{T}_{\rm Gy}$, $\tilde{\phi} = \phi(\overline{\mathbf{X}}+\overline{\boldsymbol{\rho}}) - \langle\phi(\overline{\mathbf{X}}+\overline{\boldsymbol{\rho}})\rangle$ is the gyrophase dependent part of the electrostatic potential, $\langle\cdot\rangle$ denotes the gyrophase average, ϵ_δ is the small parameter for the amplitude of ϕ and $\overline{\boldsymbol{\rho}} = \boldsymbol{\rho}(\overline{\mathbf{Z}})$. It is noted that generally $\mathsf{T}_{\rm Gy}^*\overline{F}$ is expressed as $\overline{F} + \epsilon_\delta\{S_1, \overline{F}\} + O(\epsilon_\delta^2)$ with $S_1 = (q/\Omega)\int\tilde{\phi}d\overline{\xi}$. The phase space transformations in the standard gyrokinetic formulation [5, 17] are summarised in Fig. 1. The particle phase space is transformed to the gyro-centre phase space through two steps. First, the transformation from the particle phase space to the guiding-centre phase space is performed with $\epsilon \sim \rho/L$ to remove gyrophase dependence from the single particle motion in an equilibrium magnetic field. After that, a small perturbation of electrostatic potential, ϕ, is introduced and the transformation from the guiding-centre phase space to the gyro-centre phase space is performed with $\epsilon_\delta \sim q\phi/T$ to remove the gyrophase dependence reintroduced by ϕ. Thus smallness of ϕ is necessary for the gyro-centre transformation. Push-forward of a scalar function is shown in Fig. 2 schematically. Here we consider a scalar function on $\mathbf{z}$ denoted by f and a transformation from the particle $\mathbf{z}$ space to the guiding-centre $\mathbf{Z}$ space denoted by $\mathsf{T}_{\rm GC}$. Then we can represent $f(\mathbf{z})$ in terms of $\mathbf{Z}$ through $\mathbf{z} = \mathsf{T}_{\rm GC}^{-1}\mathbf{Z}$ as $f(\mathbf{z}) = f(\mathsf{T}_{\rm GC}^{-1}\mathbf{Z}) = \mathsf{T}_{\rm GC}^{-1*}f(\mathbf{Z})$. Thus we obtain a function $F \equiv \mathsf{T}_{\rm GC}^{-1*}f$ on $\mathbf{Z}$. Since $\mathsf{T}_{\rm GC}^{-1*}$ "pushes forward" f on $\mathbf{z}$ to F on $\mathbf{Z}$, it is called the push-forward transformation associated with $\mathsf{T}_{\rm GC}$. Note that the function of the transformation is opposite to appearance of the symbol. Conversely, we can obtain a function on $\mathbf{z}$, $\mathsf{T}_{\rm GC}^*G$, from a function on $\mathbf{Z}$ denoted by G. Then $\mathsf{T}_{\rm GC}^*$ is called the pull-back transformation associated with $\mathsf{T}_{\rm GC}$. Some explanation would be needed for $\mathsf{T}_{\rm GC}^{-1}\overline{\mathbf{X}}$. Originally $[\mathsf{T}_{\rm GC}^{-1}\mathbf{x}](\mathbf{Z}) = \mathbf{X} + \boldsymbol{\rho}(\mathbf{Z}) + \cdots$

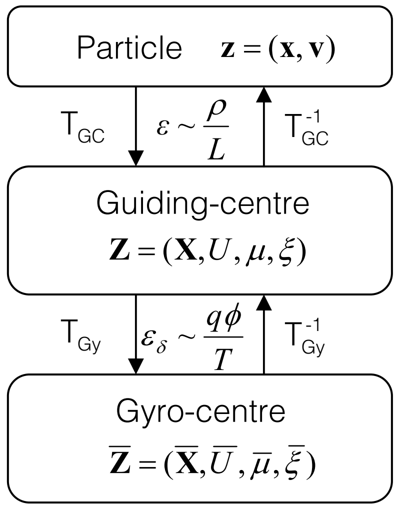

Fig. 1 Phase space transformations in the standard gyrokinetic formulation.

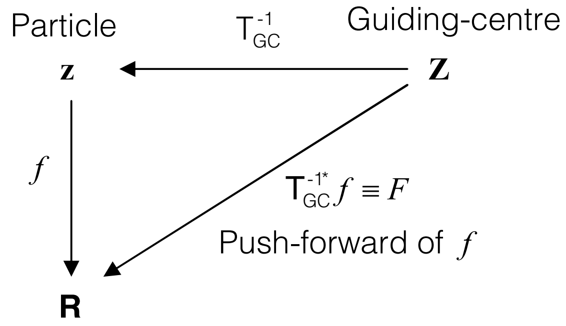

Fig. 2 Push-forward of a scalar function. $F(\mathbf{Z}) = \mathsf{T}_{\mathrm{GC}}^{-1*}f(\mathbf{Z}) = f(\mathsf{T}_{\mathrm{GC}}^{-1}\mathbf{Z}) = f(\mathbf{z})$.

denotes the particle position in the guiding-centre phase space. $\mathsf{T}_{\mathrm{GC}}^{-1}\bar{\mathbf{X}}$ denotes $[\mathsf{T}_{\mathrm{GC}}^{-1}\mathbf{x}]$ whose argument is replaced by the gyro-centre coordinates $\bar{\mathbf{Z}}$, that is, $\mathsf{T}_{\mathrm{GC}}^{-1}\bar{\mathbf{X}} \simeq \bar{\mathbf{X}}+\rho(\bar{\mathbf{Z}})$ and does not agree with the particle position in the gyro-centre phase space $\mathsf{T}_{\mathrm{Gy}}^{-1}\mathsf{T}_{\mathrm{GC}}^{-1}\mathbf{x}$. Expanding the delta function and ϕ in powers of $\bar{\rho}$ and integrating by parts lead to the representation of the particle fluid moment m_{kl} in terms of gyrofluid moments up to $O(\epsilon^2)$ (see appendix A),

$$\begin{aligned} m_{kl}(\mathbf{r}) &= M_{kl}(\mathbf{r}) + \frac{1}{2}\nabla\cdot\left[\frac{1}{q\Omega}\nabla_\perp M_{k+1l}(\mathbf{r})\right] \\ &\quad +(k+1)\nabla\cdot\left[\frac{M_{kl}(\mathbf{r})}{B_0\Omega}\nabla_\perp\phi(\mathbf{r})\right], \end{aligned} \tag{13}$$

where M_{kl} is the general gyrofluid moment defined by

$$M_{kl}(\bar{\mathbf{X}}) \equiv \int \bar{\mu}^k\bar{U}^l\mathcal{J}(\bar{\mathbf{Z}})\bar{F}(\bar{\mathbf{Z}})d\bar{U}d\bar{\mu}d\bar{\xi}, \tag{14}$$

and we have assumed that $(k_\perp\rho)^2 \sim \epsilon_\perp$ for a small $O(\epsilon_\delta)$ perturbation, $(k_\perp\rho) \sim \epsilon_\perp$ for a $O(1)$ moment and $\epsilon \sim \epsilon_\delta \sim \epsilon_\perp$. For $k = l = 0$, we have the well known push-forward representation of the particle density n,

$$n = N + \nabla\cdot\left[\frac{N}{\Omega B_0}\nabla_\perp\phi\right] + \frac{1}{2}\nabla\cdot\left[\frac{\nabla_\perp P_\perp}{q\Omega B_0}\right], \tag{15}$$

where N and $P_\perp$ are the gyro-centre density and the gyro-centre perpendicular pressure defined by

$$N \equiv \int \bar{F}\mathcal{J}d\bar{U}d\bar{\mu}d\bar{\xi}, \tag{16}$$

$$P_\perp \equiv \int \bar{\mu}B_0\bar{F}\mathcal{J}d\bar{U}d\bar{\mu}d\bar{\xi}, \tag{17}$$

respectively. Although spatial derivatives are reduced to the perpendicular Laplacian in ref. [3], we keep the divergence form to manifest the polarisation effects here [5, 8]. It is easily shown by the volume integration of eq. (15) over the domain including the whole plasma that the number of particles included in the whole plasma agrees with that of gyro-centres.

3.2 Modified guiding-centre case

In this subsection we consider a particle fluid moment defined by

$$\bar{m}_{kl}(\mathbf{r}) \equiv \int\left(\frac{mw^2}{2B_0}\right)^k v_\parallel^l f\delta^3(\mathbf{x}-\mathbf{r})d^3\mathbf{x}d^3\mathbf{v}. \tag{18}$$

Note that $w = |\mathbf{v}_\perp - \mathbf{D}|$ is used for the definition of $\bar{m}_{kl}$, while $v_\perp$ is used for m_{kl}. Therefore, $\bar{m}_{kl}$ with $k \neq 0$ is different from m_{kl}. For example, the relation between $p_\perp \equiv B_0\bar{m}_{10}$ and $p'_\perp \equiv B_0 m_{10}$ is given by

$$p_\perp = p'_\perp - mn\mathbf{V}_\perp\cdot\mathbf{D} + mnD^2/2, \tag{19}$$

where

$$n\mathbf{V}_\perp \equiv \int \mathbf{v}_\perp f d^3\mathbf{v}. \tag{20}$$

Since $\mathbf{V}_\perp = \mathbf{D} + O(\epsilon)$ in the strong $\mathbf{E}\times\mathbf{B}$ case, the relation between $p_\perp$ and $p'_\perp$ is written as

$$p_\perp = p'_\perp - mnD^2/2 + O(\epsilon). \tag{21}$$

Obviously, $\bar{m}_{kl} \to m_{kl}$ as $\mathbf{D} \to 0$. The $\mathbf{E}\times\mathbf{B}$ drift velocity $\mathbf{D}$ is sub-thermal in most cases of interest. Therefore, we assume $\mathbf{D} \sim \epsilon^{1/2}v_{ti}$ in the following. In terms of the modified guiding-centre distribution function F, $\bar{m}_{kl}$ can be written as

$$\begin{aligned} \bar{m}_{kl}(\mathbf{r}) &= \int d^6\mathbf{Z}\mathcal{J}(\mathbf{Z}) \\ &\quad \times\left[\mathsf{T}_{\mathrm{GC}}^{-1*}\left\{\left(\frac{mw^2}{2B_0}\right)^k(u)^l\right\}\right](\mathbf{Z}) \\ &\quad \times F(\mathbf{Z})\delta^3(\mathsf{T}_{\mathrm{GC}}^{-1}\mathbf{x}-\mathbf{r}) \\ &\simeq \int d^6\mathbf{Z}\mathcal{J}(\mathbf{Z})(\mu - G_1^\mu)^k U^l F(\mathbf{Z}) \\ &\quad \delta^3(\mathbf{X}+\boldsymbol{\rho}+\boldsymbol{\rho}_E-\mathbf{r}). \end{aligned} \tag{22}$$

Here the μ component of the vector field generating the guiding-centre transformation, G_1^μ, is kept because it has the nonvanishing gyroaveraged part due to the $\mathbf{E}\times\mathbf{B}$ flow. The delta function expanded in powers of $\boldsymbol{\rho}$ and $\boldsymbol{\rho}_E$, the above equation is rewritten as

$$\begin{aligned} \bar{m}_{kl}(\mathbf{r}) &\simeq \int d^6\mathbf{Z}\mathcal{J}(\mu^k - k\mu^{k-1}G_1^\mu)U^l F(\mathbf{Z}) \\ &\quad \times\Big[\delta^3(\mathbf{X}-\mathbf{r}) + \boldsymbol{\rho}_E\cdot\nabla\delta^3(\mathbf{X}-\mathbf{r}) \\ &\quad +\frac{1}{2}(\boldsymbol{\rho}\cdot\nabla)^2\delta^3(\mathbf{X}-\mathbf{r})\Big] \\ &\simeq \int d^6\mathbf{Z}\delta^3(\mathbf{X}-\mathbf{r})\mu^k U^l F\mathcal{J} \\ &\quad -k\int d^6\mathbf{Z}\delta^3(\mathbf{X}-\mathbf{r})\mu^{k-1}\langle G_1^\mu\rangle U^l F\mathcal{J} \\ &\quad +\int d^6\mathbf{Z}\mu^k U^l F\mathcal{J}\boldsymbol{\rho}_E\cdot\nabla\delta^3(\mathbf{X}-\mathbf{r}) \\ &\quad +\int d^6\mathbf{Z}\mu^k U^l F\mathcal{J}\frac{1}{2}(\boldsymbol{\rho}\cdot\nabla)^2\delta^3(\mathbf{X}-\mathbf{r}). \end{aligned} \tag{23}$$

Since the main part of $\langle G_1^\mu\rangle$ is given by [11]

$$\begin{aligned}\langle G_1^\mu\rangle &\simeq -\frac{\mu}{\Omega}\hat{b}\cdot\nabla\times(\mathbf{D}+U\hat{b})\\ &\simeq -\nabla\cdot\left(\mathbf{D}\times\frac{\mu}{\Omega}\hat{b}\right)\\ &= -\nabla\cdot\left(\frac{\mu}{\Omega B_0}\nabla_\perp\phi\right),\end{aligned}\tag{24}$$

the second term on the right hand side of eq. (23) becomes

$$\begin{aligned}&-k\int d^6\mathbf{Z}\mu^{k-1}\langle G_1^\mu\rangle U^l F\mathcal{J}\delta^3(\mathbf{X}-\mathbf{r})\\ &\simeq k\int d^6\mathbf{Z}\delta^3(\mathbf{X}-\mathbf{r})\mu^k U^l\\ &\quad\times\left[\nabla\cdot\left(\mathbf{D}\times\frac{F\mathcal{J}}{\Omega}\hat{b}\right)-\mathbf{D}\times\frac{\hat{b}}{\Omega}\cdot\nabla F\mathcal{J}\right]\\ &\simeq k\int d^6\mathbf{Z}\delta^3(\mathbf{X}-\mathbf{r})\left[\nabla\cdot\left(\frac{\mu^kU^l}{\Omega B_0}F\mathcal{J}\nabla_\perp\phi\right)\right.\\ &\quad\left.-\mathbf{D}\cdot\frac{\hat{b}\times\nabla F\mathcal{J}\mu^kU^l}{\Omega}\right]\\ &= k\nabla\cdot\left(\frac{M_{kl}}{\Omega B_0}\nabla_\perp\phi\right)-k\mathbf{D}\cdot\frac{\hat{b}\times\nabla M_{kl}}{\Omega},\end{aligned}\tag{25}$$

where the guiding-centre fluid moment M_{kl} is defined by

$$M_{kl}(\mathbf{X})\equiv\int\mu^kU^l\mathcal{J}(\mathbf{Z})F(\mathbf{Z})dUd\mu d\xi.\tag{26}$$

Integrating by parts the third term on the right hand side of eq. (23), we have

$$\begin{aligned}&\int d^6\mathbf{Z}\mu^kU^lF\mathcal{J}\boldsymbol{\rho}_E\cdot\nabla\delta^3(\mathbf{X}-\mathbf{r})\\ &= -\int d^6\mathbf{Z}\delta^3(\mathbf{X}-\mathbf{r})\nabla\cdot\left(\mu^kU^lF\mathcal{J}\frac{\hat{b}\times\mathbf{D}}{\Omega}\right)\\ &= \int d^6\mathbf{Z}\delta^3(\mathbf{X}-\mathbf{r})\nabla\cdot\left(\mu^kU^lF\mathcal{J}\frac{\nabla_\perp\phi}{\Omega B_0}\right)\\ &= \nabla\cdot\left(\frac{M_{kl}}{\Omega B_0}\nabla_\perp\phi\right).\end{aligned}\tag{27}$$

The first and last terms on the right hand side of eq. (23) are the same as those in the standard gyrokinetic model. As a consequence, the push-forward representation of $\overline{m}_{kl}$ in terms of the modified guiding-centre fluid moments is given by

$$\begin{aligned}\overline{m}_{kl}(\mathbf{r}) &= M_{kl}(\mathbf{r})+\frac{1}{2}\nabla\cdot\left[\frac{1}{q\Omega}\nabla_\perp M_{k+1l}(\mathbf{r})\right]\\ &\quad+(k+1)\nabla\cdot\left[\frac{M_{kl}(\mathbf{r})}{B_0\Omega}\nabla_\perp\phi(\mathbf{r})\right]\\ &\quad-k\mathbf{D}(\mathbf{r})\cdot\frac{\hat{b}\times\nabla M_{kl}(\mathbf{r})}{\Omega}.\end{aligned}\tag{28}$$

This representation is the same as eq. (13) except the last term formally. The last term goes to the higher order than $O(\epsilon^2)$ in the same ordering used in deriving eq. (13). Then eq. (28) agrees with eq. (13) up to $O(\epsilon^2)$. The last term has no effect for $k=0$. For $k=l=0$, therefore, we have

$$n = N+\nabla\cdot\left[\frac{N}{\Omega B_0}\nabla_\perp\phi\right]+\frac{1}{2}\nabla\cdot\left[\frac{\nabla_\perp P_\perp}{q\Omega B_0}\right],\tag{29}$$

where N and $P_\perp$ are the guiding-centre density and the guiding-centre perpendicular pressure defined by

$$N\equiv\int F\mathcal{J}dUd\mu d\xi,\tag{30}$$

$$P_\perp\equiv\int\mu B_0F\mathcal{J}dUd\mu d\xi,\tag{31}$$

respectively. Equation (29) is the same as the standard gyrokinetic case (15) formally.

3.3 Variational derivation of the push-forward representation for the particle density

The push-forward representation for the particle density is also derived from the functional derivative of the action functional $I=\int_{t_1}^{t_2}Ldt$ with the Lagrangian for the Vlasov-Poisson system,

$$\begin{aligned}L &= \sum_s\int d^6\mathbf{Z}'\mathcal{J}_s(\mathbf{Z}')F_s(\mathbf{Z}',t')\\ &\quad\times L_s[\mathbf{Z}_s(\mathbf{Z}',t';t),\dot{\mathbf{Z}}_s(\mathbf{Z}',t';t),t]\\ &\quad-\int d^3\mathbf{x}\frac{1}{4\mu_0}\mathsf{F}:\mathsf{F},\end{aligned}\tag{32}$$

where L_s is the single particle Lagrangian of species s obtained from the 1-form eq. (5),

$$L_s = q_s\mathbf{A}_s^*\cdot\dot{\mathbf{X}}_s+\frac{m_s}{q_s}\mu_s\dot{\xi}_s-H_s,\tag{33}$$

$\mathbf{Z}'\equiv(\mathbf{X}',U',\mu',\xi')$ and $\mathbf{Z}_s(\mathbf{Z}',t';t)$ denotes the guiding-centre coordinates of the particle at t with the initial condition, $\mathbf{Z}_s(\mathbf{Z}',t';t')=\mathbf{Z}'$. The guiding-centre Hamiltonian H_s is given by eq. (6). The last part of L is the Lagrangian for the electromagnetic fields in which μ_0 is permeability of vacuum, the electromagnetic field tensor F is defined by $\mathsf{F}_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu$ and $\mathsf{F}:\mathsf{F}\equiv\mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}$. When $\eta_{\mu\nu}=\eta^{\mu\nu}=\mathrm{diag}(-1,+1,+1,+1)$ is taken as a metric tensor of Minkowski spacetime, the covariant four vector potential and the four gradient operator are $A_\mu=(-\phi/c,\mathbf{A}_0)$ and $\partial_\mu=((1/c)\partial t,\nabla)$, respectively [8,18]. From $\delta I/\delta\phi=0$, we obtain the guiding-centre Poisson equation [11],

$$\begin{aligned}\epsilon_0\nabla^2\phi(\mathbf{x},t) &= -\sum_s\left[q_sN_s+q_s\nabla\cdot\left(N_s\frac{\nabla_\perp\phi}{B_0\Omega_s}\right)\right.\\ &\quad\left.-\nabla\cdot\left\{\left(\nabla\times P_{\perp s}\frac{\hat{b}}{B_0}\right)\times\frac{\hat{b}}{2\Omega_s}\right\}\right]\\ &= -\sum_s q_sn_s,\end{aligned}\tag{34}$$

where ϵ_0 is permittivity of vacuum. It is noted that we have assumed $D\sim\epsilon^{1/2}v_{ti}$ and $(m^2D^2/4qB_0)\hat{b}\cdot\nabla\times\mathbf{D}$ in the Hamiltonian (6) has been neglected. The right hand side of the above equation is the sum of charge density of species s. Noting

$$\begin{aligned}\left(\nabla\times P_\perp\frac{\hat{b}}{B_0}\right)\times\frac{\hat{b}}{2\Omega} &\simeq \left(\nabla P_\perp\times\frac{\hat{b}}{B_0}\right)\times\frac{\hat{b}}{2\Omega}\\ &\simeq -\frac{\nabla_\perp P_\perp}{2B_0\Omega},\end{aligned}\tag{35}$$

we have the same relation between the particle density n_s and the guiding-centre fluid moments as eq. (29).

4. Summary and discussion

In the previous section it was shown that the push-forward representation of the particle fluid moment in terms of the modified guiding-centre fluid moments is very similar to that in terms of the standard gyrofluid moments in the appropriate limit. However, there is an apparent difference between the exact representations (12) and (22). Equation (22) has the standard form of the push-forward representation associated with the guiding-centre trnasformation, while eq. (12) is the mixed representation of the push-forward transformation associated with $\mathsf{T}_{\rm GC}$ and the pull-back transformation assciated with $\mathsf{T}_{\rm Gy}$. The representation (12) is derived by considering $\mathbf{Z}$ in the integral on the right hand side of eq. (22) a dummy variable, replacing $\mathbf{Z}$ by $\bar{\mathbf{Z}}$ and using $F = \mathsf{T}^*_{\rm Gy}\bar{F}$ [17]. Since the perturbation of the electrostatic potential is introduced after the guiding-centre transformation in the standard gyrokinetic formulation, its effect is not included in $\mathsf{T}_{\rm GC}$, but in $\mathsf{T}_{\rm Gy}$. Therefore the effect of ϕ is included only in $\mathsf{T}^*_{\rm Gy}\bar{F}$ in eq. (12). On the other hand, it is included in $\mathsf{T}_{\rm GC}$ in the modified guiding-centre case. The correspondence between the standard gyrokinetic case and the modified guiding-centre case becomes more transparent by considering the alternative form similar to eq. (22) given by [19]

$$
\begin{aligned}
m_{kl}(\mathbf{r}) &= \int d^6\bar{\mathbf{Z}}\mathcal{J}(\bar{\mathbf{Z}}) \\
&\quad\times\left[\mathsf{T}^{-1*}_{\rm Gy}\mathsf{T}^{-1*}_{\rm GC}\left\{\left(\frac{mv_\perp^2}{2B_0}\right)^k v_\parallel^l\right\}\right] \\
&\quad\times\bar{F}(\bar{\mathbf{Z}})\delta^3(\mathsf{T}^{-1}_{\rm Gy}\mathsf{T}^{-1}_{\rm GC}\mathbf{x}-\mathbf{r}). \qquad (36)
\end{aligned}
$$

This is more straightforward representation than eq. (12). The particle position in the gyro-centre phase space $\mathsf{T}^{-1}_{\rm Gy}\mathsf{T}^{-1}_{\rm GC}\mathbf{x}$ is given by

$$
\mathsf{T}^{-1}_{\rm Gy}\mathsf{T}^{-1}_{\rm GC}\mathbf{x} = \bar{\mathbf{X}} + \bar{\boldsymbol{\rho}} + \bar{\boldsymbol{\rho}}_1 + \cdots, \qquad (37)
$$

where $\bar{\boldsymbol{\rho}}_1 = -\{S_1, \bar{\mathbf{X}} + \bar{\boldsymbol{\rho}}\}$ is the gyro-centre displacement vector [5, 6]. The gyroangle average of $\bar{\boldsymbol{\rho}}_1$ corresponds to $\boldsymbol{\rho}_E$ in the small amplitude limit. Since the standard guiding-centre magnetic moment μ is related with the gyro-centre coordinates as

$$
\begin{aligned}
\mu &\simeq \bar{\mu} - \{S_1, \bar{\mu}\} \\
&\simeq \bar{\mu} - \frac{\partial S_1}{\partial \bar{\xi}}\{\bar{\xi}, \bar{\mu}\} \\
&\simeq \bar{\mu} - \frac{q}{\Omega}\tilde{\phi}(\bar{\mathbf{X}} + \bar{\boldsymbol{\rho}})\frac{q}{m}, \qquad (38)
\end{aligned}
$$

we have

$$
\begin{aligned}
\mathsf{T}^{-1*}_{\rm Gy}\mathsf{T}^{-1*}_{\rm GC}\left(\frac{mv_\perp^2}{2B_0}\right) &\simeq \mu(\bar{\mathbf{Z}}) - G_1^\mu(\bar{\mathbf{Z}}) \\
&\simeq \bar{\mu} - \frac{q}{B_0}\tilde{\phi}(\bar{\mathbf{X}} + \bar{\boldsymbol{\rho}}) - G_1^\mu(\bar{\mathbf{Z}}). \qquad (39)
\end{aligned}
$$

Using the above result and expanding the delta function and ϕ, we have

$$
\begin{aligned}
m_{kl}(\mathbf{r}) &\simeq \int d^6\bar{\mathbf{Z}}\mathcal{J}\left[\bar{\mu}^k - k\bar{\mu}^{k-1}\frac{q}{B_0}\tilde{\phi}(\bar{\mathbf{X}} + \bar{\boldsymbol{\rho}})\right] \\
&\quad\times\bar{U}^l\bar{F}\delta^3(\bar{\mathbf{X}} + \bar{\boldsymbol{\rho}} + \bar{\boldsymbol{\rho}}_1 - \mathbf{r}) \\
&\simeq \int d^6\bar{\mathbf{Z}}\mathcal{J}\bar{F}\bar{\mu}^k\bar{U}^l\Big[\delta^3(\bar{\mathbf{X}} - \mathbf{r}) \\
&\quad+\langle\bar{\boldsymbol{\rho}}_1\rangle\cdot\nabla\delta^3(\bar{\mathbf{X}} - \mathbf{r}) + \frac{1}{2}(\bar{\boldsymbol{\rho}}\cdot\nabla)^2\delta^3(\bar{\mathbf{X}} - \mathbf{r})\Big] \\
&\quad-k\int d^6\bar{\mathbf{Z}}\mathcal{J}\bar{F}\bar{\mu}^{k-1}\bar{U}^l\frac{q}{B_0}\phi(\bar{\mathbf{X}} + \bar{\boldsymbol{\rho}}) \\
&\quad\times\bar{\boldsymbol{\rho}}\cdot\nabla\delta^3(\bar{\mathbf{X}} - \mathbf{r}) \\
&\simeq M_{kl} + \nabla\cdot\left(\frac{M_{kl}}{\Omega B_0}\nabla_\perp\phi\right) + \frac{1}{2}\nabla\cdot\left(\frac{\nabla_\perp M_{k+1l}}{q\Omega}\right) \\
&\quad+k\nabla\cdot\left(\frac{M_{kl}}{\Omega B_0}\nabla_\perp\phi\right), \qquad (40)
\end{aligned}
$$

where the contribution from G_1^μ has been dropped because in the standard guiding-centre theory $\langle G_1^\mu\rangle = -\mu U\hat{b}\cdot\nabla\times\hat{b}/\Omega$ and variation of the magnetic field has been assumed to be very mild, and the gyrofliud moment is defined by eq. (14). Thus the alternative form of the push-forward representation yields the same representation as eq. (13) and gives the direct correspondence to the representation in the modified guiding-centre case.

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A. Derivation of the push-forward representation in the standard gyrokinetic model

The first term on the right hand side of eq. (12) yields

$$
\begin{aligned}
&\int d^6\bar{\mathbf{Z}}\bar{\mu}^k\bar{U}^l\bar{F}(\bar{\mathbf{Z}})\delta^3(\bar{\mathbf{X}} + \bar{\boldsymbol{\rho}} - \mathbf{r}) \\
&= \int d^6\bar{\mathbf{Z}}\bar{\mu}^k\bar{U}^l\bar{F}(\bar{\mathbf{Z}})\Big[\delta^3(\bar{\mathbf{X}} - \mathbf{r}) \\
&\quad+\frac{1}{2}(\bar{\boldsymbol{\rho}}\cdot\bar{\nabla})^2\delta^3(\bar{\mathbf{X}} - \mathbf{r}) + \cdots\Big] \\
&\simeq M_{kl}(\mathbf{r}) + \frac{1}{2}\nabla\cdot\left[\frac{1}{q\Omega}\nabla_\perp M_{k+1l}(\mathbf{r})\right]. \qquad (41)
\end{aligned}
$$

The second term of eq. (12) becomes

$$
\begin{aligned}
&\int d^6\bar{\mathbf{Z}}\bar{\mu}^k\bar{U}^l\frac{q\tilde{\phi}}{B_0}\frac{\partial\bar{F}}{\partial\bar{\mu}}\delta^3(\bar{\mathbf{X}} + \bar{\boldsymbol{\rho}} - \mathbf{r}) \\
&\simeq \int d^6\bar{\mathbf{Z}}\bar{\mu}^k\bar{U}^l\frac{q}{B_0}\bar{\boldsymbol{\rho}}\cdot\bar{\nabla}\phi(\bar{\mathbf{X}})\frac{\partial\bar{F}}{\partial\bar{\mu}}\bar{\boldsymbol{\rho}}\cdot\bar{\nabla}\delta^3(\bar{\mathbf{X}} - \mathbf{r}) \\
&= \int d^6\bar{\mathbf{Z}}\delta^3(\bar{\mathbf{X}} - \mathbf{r}) \\
&\quad\times\bar{\nabla}\cdot\left[(k+1)\bar{\mu}^k\bar{U}^l\frac{\bar{F}}{B_0\Omega}\bar{\nabla}_\perp\phi(\bar{\mathbf{X}})\right] \\
&= (k+1)\nabla\cdot\left[\frac{M_{kl}(\mathbf{r})}{B_0\Omega}\nabla_\perp\phi(\mathbf{r})\right]. \qquad (42)
\end{aligned}
$$

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