J. Plasma Fusion Res. SERIES, Vol. 9 (2010)

Action-angle representation of waves and instabilities in flowing plasmas

Makoto HIROTA

Japan Atomic Energy Agency, Naka, Ibaraki, 311-0193, Japan

(Received: 27 October 2009 / Accepted: 17 February 2010)

Action-angle representation of linear fluctuation in plasma is formally performed by taking the Lagrange-Hamilton theory as the basis and invoking the recent spectral technique [Hirota and Fukumoto, J. Math. Phys. **49** 083101 (2008)], which is shown to be applicable equally to the magnetohydrodynamic (MHD) system and the Vlasov-Maxwell system. This formalism leads to a natural definition of the wave energy (as well as the wave action) for each eigenmode and continuum mode. Negative energy mode is generally responsible for onsets of various instabilities in flowing plasmas. It is shown that the canonical form of the linearized system, formulated by Hamilton's principle of least action, facilitates the consideration of the action-angle representation, and the spectral technique becomes more sophisticated than the previous work that based on the noncanonical form.

Keywords: action-angle variables, flowing plasmas, magnetohydrodynamics, Vlasov-Maxwell equation, continuous spectrum

1. Background

Stabilization or destabilization of plasma by flow is a key issue in recent fusion research and astrophysics. While its mechanism is drawing a lot of attention, theories for flowing plasmas are quite limited in comparison to static plasmas; especially, clear-cut stability criteria are hard to obtain. For example, the variational approaches[1, 2, 3, 4], which are nowadays well-known such as the energy principle in the MHD theory [5], the Arnold method [6, 7] and the energy-Casimir method [8], are often fruitless for flowing plasmas, i.e., they end up with extremely limited stability conditions especially for 3D disturbances. In order to gain a better understanding of various instabilities in flowing plasmas, it is informative to study the energy of eigenmode in the context of the stability theory of Hamiltonian system [9, 10, 11, 12], in which negative energy mode is generally regarded as a source of instability. It is well known that negative energy mode may occur in moving medium (i.e., flow) [13], which is not always unstable but tend to be destabilized in combination with positive energy modes (reactive instability [9, 14, 15]) or by small dissipation effect (dissipation-induced instability [16]).

However, such Hamiltonian theories are not yet matured for the "continuum mode" that is well-known to occur in fluctuation of plasma. The existence of continuum mode (or continuous spectrum) requires careful mathematical treatments of singularities in the eigenvalue problem. The wave energy for continuous spectrum was first successfully derived by Morrison and Pfirsch [17] in the linear Vlasov-Poisson system, where the corresponding action-angle representation is rigorously performed by means of a singular integral transform akin to the Hilbert transform [18]. Recently, we have developed another new spectral technique [19], which enables us to evaluate wave energy (and also action-angle variables) for each eigenmode and continuum mode in a unified manner by invoking the Laplace transform. Our method has been applied to the vortical continuum mode in shear flow [19] and the Alfvén and sound continua [20] in the MHD case. The action-angle representation of linear fluctuation serves to provide a Hamiltonian interpretation of various resonant instabilities in flowing plasmas. For instance, it is shown [19] that resonant coupling between an eigenmode and a continuum mode having the same sign of energy results in phase mixing (or continuum) damping. In contrast, such resonance triggers an instability if their signs of energy are opposite. The stabilization of the resistive wall mode by flow can be discussed in this general framework [21].

In this paper, we will rework our spectral method [19] in a more sophisticated way by the use of the Lagrangian approach to plasmas [22, 23, 24, 25]. We first introduce Hamilton's principle of least action into linear perturbation, which enables us to write the linearized system in the canonical form. This procedure will be demonstrated for the ideal MHD equation and the Vlasov-Maxwell equation as well, from which one can expect the generalizations to other equations. The canonical form is more intuitive and tractable than the noncanonical form which we have employed in the previous work [19]. By constructing an appropriate Poincaré invariant, we will effectively perform the action-angle representation even in the presence of growing/damping eigenmodes and continuum modes, which are not periodic in time by nature. The detailed calculation of wave energy for eigenmodes and

author's e-mail: hirota.makoto@jaea.go.jp

continuum modes has been done for the MHD case by assuming a slab geometry [20]. The present work indicates that the same treatment is also valid for the Vlasov-Maxwell system.

2. Hamilton's principle of least action for linearized systems

In this section, we discuss the canonical aspect of linear perturbation according to the Lagrange-Hamilton theory. In order to arrive at the canonical form of a linearized dynamical system, it is essential to note that the linearized system naturally inherits the variational principle of least action. First, let us simply review this fact by considering an action integral

$$S = \int_{t_1}^{t_2} L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)dt, \tag{1}$$

with a Lagrangian L for generalized coordinates $\boldsymbol{q}(t) = (q_1, q_2, \ldots, q_n)(t)$ and velocities $\dot{\boldsymbol{q}} = d\boldsymbol{q}/dt$. Hamilton's principle of least action $\delta S = 0$ yields the Euler-Lagrange equation for $\boldsymbol{q}(t)$,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) = \frac{\partial L}{\partial \boldsymbol{q}} \tag{2}$$

Now, suppose that $\boldsymbol{q}(t)$ is any given solution (including steady one $d\boldsymbol{q}/dt = 0$). We try searching a perturbed solution $\boldsymbol{q}'(t) = \boldsymbol{q}(t) + \tilde{\boldsymbol{q}}(t)$ in accordance with Hamilton's principle. By substituting the trial function $\boldsymbol{q}'(t)$ into S, we obtain the following expansion with respect to the perturbation $\tilde{\boldsymbol{q}}(t)$,

$$S = S^{(0)} + S^{(1)} + S^{(2)} + S^{(3)} + \ldots \tag{3}$$

Since the basic solution $\boldsymbol{q}(t)$ is already extremum $\delta S = 0$, we find that $S^{(1)} = 0$ holds automatically. If the amplitude of $\tilde{\boldsymbol{q}}(t)$ is sufficiently small compared with $\boldsymbol{q}(t)$, one may neglect the higher order terms $S^{(n)}$, $n \geq 3$, and obtain an action integral for the perturbation $\tilde{\boldsymbol{q}}(t)$ as

$$\begin{aligned} S^{(2)} &= \int_{t_1}^{t_2} L^{(2)} dt \qquad (4)\\ &= \int_{t_1}^{t_2} \frac{1}{2}\Bigg(\dot{\tilde{\boldsymbol{q}}} \cdot \frac{\partial^2 L}{\partial \dot{\boldsymbol{q}}^2}\dot{\tilde{\boldsymbol{q}}} + 2\tilde{\boldsymbol{q}} \cdot \frac{\partial^2 L}{\partial \boldsymbol{q}\partial \dot{\boldsymbol{q}}}\dot{\tilde{\boldsymbol{q}}} \\ &\qquad + \tilde{\boldsymbol{q}} \cdot \frac{\partial^2 L}{\partial \boldsymbol{q}^2}\tilde{\boldsymbol{q}}\Bigg)dt, \qquad (5) \end{aligned}$$

where $L^{(2)}$ is thought to be a Lagrangian for the perturbation $\tilde{\boldsymbol{q}}(t)$ that is to be varied. Since $S^{(2)}$ is quadratic, the extremum condition $\delta S^{(2)} = 0$ with respect to $\delta\tilde{\boldsymbol{q}}$ yields the linearized Euler-Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{\boldsymbol{q}}^2}\dot{\tilde{\boldsymbol{q}}} + \frac{\partial^2 L}{\partial \dot{\boldsymbol{q}}\partial \boldsymbol{q}}\tilde{\boldsymbol{q}}\right) = \frac{\partial^2 L}{\partial \boldsymbol{q}\partial \dot{\boldsymbol{q}}}\dot{\tilde{\boldsymbol{q}}} + \frac{\partial^2 L}{\partial \boldsymbol{q}^2}\tilde{\boldsymbol{q}}. \tag{6}$$

Hamilton's principle is therefore valid even for linear dynamics. The linearized equation derived in this way is especially called the Jacobi equation [6, 7].

As usual, the Legendre transform $\tilde{\boldsymbol{p}} = \partial L^{(2)}/\partial\dot{\tilde{\boldsymbol{q}}}$ determines a quadratic Hamiltonian function $H^{(2)}$ in the phase space $(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}})$ as far as $(\partial^2 L/\partial\dot{\boldsymbol{q}}^2)^{-1}$ exists. For later use, we introduce the following unified notations;

$$\tilde{u} = \begin{pmatrix}\tilde{\boldsymbol{q}}\\ \tilde{\boldsymbol{p}}\end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \tag{7}$$

and

$$\begin{aligned} &H^{(2)}(\tilde{u}) \\ &= \frac{1}{2}\left(\tilde{\boldsymbol{p}} - \frac{\partial^2 L}{\partial\dot{\boldsymbol{q}}\partial\boldsymbol{q}}\tilde{\boldsymbol{q}}\right) \cdot \left(\frac{\partial^2 L}{\partial\dot{\boldsymbol{q}}^2}\right)^{-1}\left(\tilde{\boldsymbol{p}} - \frac{\partial^2 L}{\partial\dot{\boldsymbol{q}}\partial\boldsymbol{q}}\tilde{\boldsymbol{q}}\right) \\ &\quad - \frac{1}{2}\tilde{\boldsymbol{q}} \cdot \frac{\partial^2 L}{\partial\boldsymbol{q}^2}(\tilde{\boldsymbol{q}}) \qquad (8)\\ &=: \frac{1}{2}\langle\tilde{u}, \mathcal{H}\tilde{u}\rangle, \qquad (9) \end{aligned}$$

where we have naturally identified the phase space $\tilde{u}$ as a Hilbert space with an inner product $\langle\tilde{u}_1, \tilde{u}_2\rangle = \tilde{\boldsymbol{q}}_1 \cdot \tilde{\boldsymbol{q}}_2 + \tilde{\boldsymbol{p}}_1 \cdot \tilde{\boldsymbol{p}}_2$, and the linear operators $\mathcal{J}$ and $\mathcal{H}$ are respectively anti-symmetric and symmetric by definition. Then, the Hamiltonian equation is shortly written as

$$\partial_t\tilde{u} = \mathcal{J}\frac{\partial H^{(2)}}{\partial\tilde{u}} \quad \text{or} \quad \partial_t\tilde{u} = \mathcal{J}\mathcal{H}\tilde{u}. \tag{10}$$

It must be remarked that $H^{(2)}$ qualifies as the perturbation (or wave) energy only when the basic solution $\boldsymbol{q}(t)$ is an equilibrium state ($d\boldsymbol{q}/dt = 0$). This is because the linear term $H^{(1)}$ of the series expansion of the Hamiltonian H is not zero in general. It becomes automatically zero only at an equilibrium state where $\partial H/\partial\boldsymbol{q} = \partial H/\partial\boldsymbol{p} = 0$ is satisfied.

Let us derive the action-angle variables for an oscillatory eigenmode in the conventional manner. Suppose that the basic solution is an equilibrium state and there is a single eigenmode,

$$\tilde{u}(t) = \hat{u}e^{-i\omega t} + \text{c.c.}, \tag{11}$$

with an eigenfrequency $\omega \in \mathbb{R}$ and an eigenvector $\hat{u} = (\hat{\boldsymbol{q}}, \hat{\boldsymbol{p}})$. The complex conjugate (c.c.) is needed to insure that $\tilde{u}(t)$ is real. By introducing the angle variable as $\theta(t) = \omega t$, the action variable is represented by

$$\begin{aligned} \mu &:= \oint \tilde{\boldsymbol{p}} \cdot d\tilde{\boldsymbol{q}} = \frac{1}{2\pi}\int_0^{2\pi} \tilde{\boldsymbol{p}} \cdot \frac{\partial\tilde{\boldsymbol{q}}}{\partial\theta}d\theta \\ &= i\hat{\boldsymbol{p}} \cdot \overline{\hat{\boldsymbol{q}}} - i\overline{\hat{\boldsymbol{p}}} \cdot \hat{\boldsymbol{q}} = \left\langle\overline{\hat{u}}, i\mathcal{J}\hat{u}\right\rangle. \qquad (12) \end{aligned}$$

It is easily verified that the energy of eigenmode is $H^{(2)} = \omega\mu$.

The derivation of the canonical equation (10) that we have shown here can be applied to plasmas in a similar way. While the configuration space for the generalized coordinates is no longer finite-dimensional $\boldsymbol{q}(t) \in \mathbb{R}^n$, the Lagrange-Hamilton theory for plasma has been developed by several pioneering works. Here, we briefly summarize their results and show the canonical form explicitly.

2.1 Magnetohydrodynamics

The Lagrangian for the ideal MHD equations was clarified by Newcomb [23], where the generalized coordinates correspond to the fluid particle orbits $\boldsymbol{x}(0) \mapsto \boldsymbol{x}(t)$ that is mathematically understood as a diffeomorphism group (the Lagrangian description of fluid motion). The velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ in the Eulerian coordinates is associated with the orbits via $\dot{\boldsymbol{x}}(t) = \boldsymbol{v}(\boldsymbol{x}(t), t)$. The magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$, the mass density $\rho(\boldsymbol{x}, t)$ and the specific entropy $s(\boldsymbol{x}, t)$ are frozen to each infinitesimal fluid element convected by the flow $\boldsymbol{x}(0) \mapsto \boldsymbol{x}(t)$.

Assume that $(\boldsymbol{v}, \boldsymbol{B}, \rho, s)$ is a given basic solution of the MHD equation. The above kinematical (or topological) constraints imply that the MHD fluctuations are expressed by

$$\tilde{\boldsymbol{v}} = \partial_t \tilde{\boldsymbol{\xi}} + (\boldsymbol{v} \cdot \nabla)\tilde{\boldsymbol{\xi}} - (\tilde{\boldsymbol{\xi}} \cdot \nabla)\boldsymbol{v}, \tag{13}$$

$$\tilde{\boldsymbol{B}} = \nabla \times (\tilde{\boldsymbol{\xi}} \times \boldsymbol{B}), \tag{14}$$

$$\tilde{\rho} = -\nabla \cdot (\rho \tilde{\boldsymbol{\xi}}), \tag{15}$$

$$\tilde{s} = -\tilde{\boldsymbol{\xi}} \cdot \nabla s, \tag{16}$$

where the vector field $\tilde{\boldsymbol{\xi}}(\boldsymbol{x}, t)$ represents the displacement of the particle orbit, $\boldsymbol{x}'(t) = \boldsymbol{x}(t) + \tilde{\boldsymbol{\xi}}(\boldsymbol{x}(t), t)$, observed in the Eulerian coordinates [26, 23].

The small-amplitude expansion of the MHD Lagrangian was performed by Dewar [24], which results in

$$L^{(2)} = \int \frac{1}{2} \left[\rho \left| \frac{D\tilde{\boldsymbol{\xi}}}{Dt} \right|^2 + \tilde{\boldsymbol{\xi}} \cdot \mathcal{F}\tilde{\boldsymbol{\xi}} \right] d^3x \tag{17}$$

where $D/Dt = \partial_t + \boldsymbol{v} \cdot \nabla$. The symmetric operator $\mathcal{F}$ is defined by

$$\begin{aligned} \mathcal{F}\tilde{\boldsymbol{\xi}} =& (\boldsymbol{B} \cdot \nabla)[(\boldsymbol{B} \cdot \nabla)\tilde{\boldsymbol{\xi}}] - (\tilde{\boldsymbol{\xi}} \cdot \nabla)\nabla p_* \\ &+ \nabla[(B^2 + \rho c_s^2)\nabla \cdot \tilde{\boldsymbol{\xi}}] - (\nabla \cdot \tilde{\boldsymbol{\xi}})\nabla p_* \\ &+ \nabla(\tilde{\boldsymbol{\xi}} \cdot \nabla p_*) - (\boldsymbol{B} \cdot \nabla)[(\nabla \cdot \tilde{\boldsymbol{\xi}})\boldsymbol{B}] \\ &- \nabla\{\boldsymbol{B} \cdot [(\boldsymbol{B} \cdot \nabla)\tilde{\boldsymbol{\xi}}]\}, \end{aligned} \tag{18}$$

where $p_* = p(\rho, s) + B^2/2$ is the sum of kinetic and magnetic pressures and $c_s = \sqrt{\partial p/\partial \rho}$ is the sound speed. Hamilton's principle leads to the equation of motion,

$$\rho \frac{D^2 \tilde{\boldsymbol{\xi}}}{Dt^2} = \mathcal{F}\tilde{\boldsymbol{\xi}}, \tag{19}$$

where $D^2/Dt^2 = (\partial_t + \boldsymbol{v} \cdot \nabla)(\partial_t + \boldsymbol{v} \cdot \nabla)$ and note that ∂_t and $\boldsymbol{v} \cdot \nabla$ do not commute for a time-dependent flow $\boldsymbol{v}$. If the basic fields are an equilibrium state, this equation reduces to the one derived by Frieman and Rotenberg [26]. Together with the momentum field defined by $\tilde{\boldsymbol{m}} = \rho D\tilde{\boldsymbol{\xi}}/Dt$, (19) is written in the canonical form (10) where

$$\tilde{u} = \begin{pmatrix} \tilde{\boldsymbol{\xi}} \\ \tilde{\boldsymbol{m}} \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} -\mathcal{F} & \nabla \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \\ -\boldsymbol{v} \cdot \nabla & 1/\rho \end{pmatrix}, \tag{20}$$

and the inner product is

$$\langle \tilde{u}_1, \tilde{u}_2 \rangle = \int (\tilde{\boldsymbol{\xi}}_1 \cdot \tilde{\boldsymbol{\xi}}_2 + \tilde{\boldsymbol{m}}_1 \cdot \tilde{\boldsymbol{m}}_2) d^3x. \tag{21}$$

As was shown in (12), the action variable for an eigenmode $\tilde{\boldsymbol{\xi}}(t) = \hat{\boldsymbol{\xi}} e^{-i\omega t} + \text{c.c.}$ is calculated as

$$\begin{aligned} \mu =& i \int \left(\overline{\hat{\boldsymbol{\xi}}} \cdot \hat{\boldsymbol{m}} - \overline{\hat{\boldsymbol{m}}} \cdot \hat{\boldsymbol{\xi}} \right) d^3x \\ =& 2 \int \overline{\hat{\boldsymbol{\xi}}} \cdot \rho(\omega + i\boldsymbol{v} \cdot \nabla)\hat{\boldsymbol{\xi}} d^3x. \end{aligned} \tag{22}$$

This agrees with the spatial integration of the wave action density derived by Brizard [27].

2.2 Vlasov-Maxwell system

The Lagrangian theory for the Vlasov-Maxwell equation was formulated by Low [22]. Here, we consider a plasma consisting of one species with particle mass m and charge q (but the generalization to the multi-species case is straightforward). Let $(f, \boldsymbol{E}, \boldsymbol{B})$ be a given solution, where $f(\boldsymbol{x}, \boldsymbol{v}, t)$ is the distribution function, and $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$ are respectively the electric and magnetic fields. According to Low (see also Galloway & Kim [25]), the linear perturbation is expressed by

$$\tilde{f} = -\nabla \cdot (f\tilde{\boldsymbol{\xi}}) - \nabla_v \cdot \left(f \frac{D\tilde{\boldsymbol{\xi}}}{Dt} \right) \tag{23}$$

$$\tilde{\boldsymbol{E}} = -\nabla\tilde{\phi} - \partial_t \tilde{\boldsymbol{A}}, \tag{24}$$

$$\tilde{\boldsymbol{B}} = \nabla \times \tilde{\boldsymbol{A}} \tag{25}$$

in terms of the fluctuations of the scalar $\tilde{\phi}(\boldsymbol{x}, t)$ and vector $\tilde{\boldsymbol{A}}(\boldsymbol{x}, t)$ potentials and the displacement vector $\tilde{\boldsymbol{\xi}}(\boldsymbol{x}, \boldsymbol{v}, t)$ observed in the phase space;

$$\boldsymbol{x}'(t) = \boldsymbol{x}(t) + \tilde{\boldsymbol{\xi}}(\boldsymbol{x}(t), \boldsymbol{v}(t), t), \tag{26}$$

$$\boldsymbol{v}'(t) = \boldsymbol{v}(t) + \frac{D\tilde{\boldsymbol{\xi}}}{Dt}(\boldsymbol{x}(t), \boldsymbol{v}(t), t). \tag{27}$$

Here, one must regard the total derivative D/Dt as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla + \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \nabla_v. \tag{28}$$

The relation (23) originates from the fact that the distribution function f is frozen to the phase space flow $(\boldsymbol{x},\boldsymbol{v})(0)\mapsto(\boldsymbol{x},\boldsymbol{v})(t)$. The condition $\nabla\cdot\tilde{\boldsymbol{B}}=0$ and Faraday's law $\nabla\times\tilde{\boldsymbol{E}}+\partial_t\tilde{\boldsymbol{B}}=0$ have been already taken into account in (24) and (25). By employing the Coulomb gauge $\nabla\cdot\tilde{\boldsymbol{A}}=0$, we can eliminate the electrostatic potential $\tilde{\phi}$ from the dynamical variables since Gauss's law relates $\tilde{\phi}$ to $\tilde{\boldsymbol{\xi}}$ through the Poisson equation;

$$\epsilon_0\Delta\tilde{\phi}=-\int q\tilde{f}d^3v=\int q\nabla\cdot(f\tilde{\boldsymbol{\xi}})d^3v,\tag{29}$$

where ϵ_0 denotes the vacuum permittivity. In this way, we regard the Coulomb gauge and the Poisson equation as kinematical constraints (which are imposed on the configuration space to be varied in Hamilton's principle).

The Lagrangian for the fluctuations $\tilde{\boldsymbol{\xi}}$ and $\tilde{\boldsymbol{A}}$ is then given by

$$\begin{aligned}L^{(2)}=&\frac{1}{2}\int d^3x\int d^3v f\Bigg[m\left|\frac{D\tilde{\boldsymbol{\xi}}}{Dt}\right|^2\\&+2q[(\tilde{\boldsymbol{\xi}}\cdot\nabla)\boldsymbol{A}+\tilde{\boldsymbol{A}}]\cdot\frac{D\tilde{\boldsymbol{\xi}}}{Dt}\\&-q\tilde{\xi}_i\tilde{\xi}_j\frac{\partial^2(\phi-\boldsymbol{v}\cdot\boldsymbol{A})}{\partial x_i\partial x_j}-2q\tilde{\boldsymbol{\xi}}\cdot\nabla(\tilde{\phi}-\boldsymbol{v}\cdot\tilde{\boldsymbol{A}})\Bigg]\\&+\frac{1}{2}\int d^3x\left[\epsilon_0|\nabla\tilde{\phi}+\partial_t\tilde{\boldsymbol{A}}|^2-\frac{1}{\mu_0}|\nabla\times\tilde{\boldsymbol{A}}|^2\right]\end{aligned}\tag{30}$$

where μ_0 is the vacuum permeability. In comparison with the previous works [22, 25], note that $\tilde{\phi}$ in $L^{(2)}$ must be substituted by the solution $\tilde{\phi}$ of (29) and we have a relation,

$$\int|\nabla\tilde{\phi}+\partial_t\tilde{\boldsymbol{A}}|^2d^3x=\int(|\nabla\tilde{\phi}|^2+|\partial_t\tilde{\boldsymbol{A}}|^2)d^3x,\tag{31}$$

since the functional spaces of gradient fields and divergence-free fields are orthogonal to each other. The variation of $S^{(2)}=\int_{t_1}^{t_2}L^{(2)}dt$ with respect to $\delta\tilde{\boldsymbol{\xi}}$ and $\delta\tilde{\boldsymbol{A}}$ gives, respectively, the equation of motion and the divergence-free part of Ampère's law,

$$\begin{aligned}m\frac{D^2\tilde{\boldsymbol{\xi}}}{Dt^2}=q\tilde{\boldsymbol{\xi}}\cdot\nabla(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})+q\frac{D\tilde{\boldsymbol{\xi}}}{Dt}\times\boldsymbol{B}\\+q(\tilde{\boldsymbol{E}}+\boldsymbol{v}\times\tilde{\boldsymbol{B}}),\end{aligned}\tag{32}$$

$$\epsilon_0\partial_t^2\tilde{\boldsymbol{A}}+\frac{1}{\mu_0}\nabla\times\tilde{\boldsymbol{B}}=\mathcal{P}\left[q\int\tilde{f}\boldsymbol{v}d^3v\right],\tag{33}$$

where $\mathcal{P}$ denotes the projection onto the space of divergence-free vector fields. The gradient part of Ampère's law is automatically satisfied by the Poisson equation (29).

The corresponding momentum fields are found to be

$$\tilde{\boldsymbol{m}}=\frac{\delta L^{(2)}}{\delta\dot{\tilde{\boldsymbol{\xi}}}}=f\left[m\frac{D\tilde{\boldsymbol{\xi}}}{Dt}+q(\tilde{\boldsymbol{\xi}}\cdot\nabla)\boldsymbol{A}+q\tilde{\boldsymbol{A}}\right],\tag{34}$$

$$\tilde{\boldsymbol{Y}}=\frac{\delta L^{(2)}}{\delta\dot{\tilde{\boldsymbol{A}}}}=\epsilon_0\partial_t\tilde{\boldsymbol{A}},\tag{35}$$

where the phase space $\tilde{u}=(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{A}},\tilde{\boldsymbol{m}},\tilde{\boldsymbol{Y}})$ is constrained by $\nabla\cdot\tilde{\boldsymbol{A}}=\nabla\cdot\tilde{\boldsymbol{Y}}=0$. The Legendre transform determines a proper Hamiltonian $H^{(2)}$ and canonical equations,

$$\begin{aligned}\partial_t\tilde{\boldsymbol{\xi}}&=\frac{\delta H^{(2)}}{\delta\tilde{\boldsymbol{m}}},\quad\partial_t\tilde{\boldsymbol{m}}=-\frac{\delta H^{(2)}}{\delta\tilde{\boldsymbol{\xi}}},\\\partial_t\tilde{\boldsymbol{A}}&=\frac{\delta H^{(2)}}{\delta\tilde{\boldsymbol{Y}}},\quad\partial_t\tilde{\boldsymbol{Y}}=-\frac{\delta H^{(2)}}{\delta\tilde{\boldsymbol{A}}},\end{aligned}\tag{36}$$

which are shown explicitly in the Appendix.

For an eigenmode $(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{A}})(t)=(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{A}})e^{-i\omega t}+\text{c.c.}$, the action variable is calculated by

$$\begin{aligned}\mu=&i\int\int\left(\overline{\hat{\boldsymbol{\xi}}}\cdot\hat{\boldsymbol{m}}-\overline{\hat{\boldsymbol{m}}}\cdot\hat{\boldsymbol{\xi}}\right)d^3vd^3x\\&+i\int\left(\overline{\hat{\boldsymbol{A}}}\cdot\hat{\boldsymbol{Y}}-\overline{\hat{\boldsymbol{Y}}}\cdot\hat{\boldsymbol{A}}\right)d^3x,\end{aligned}\tag{37}$$

where $\hat{\boldsymbol{m}}$ and $\hat{\boldsymbol{Y}}$ can be eliminated by using the relations, (34) and (35) (e.g., $\hat{\boldsymbol{Y}}=-i\omega\epsilon_0\hat{\boldsymbol{A}}$).

For the purpose of finding the action-angle variables, this canonical formalism is quite beneficial and intuitive. However, in contrast to the MHD case, the equations (32) is not necessarily required to be solved in many cases, since we are usually interested in the behavior of the scalar function $\tilde{f}$, not in the two vector fields $\tilde{\boldsymbol{\xi}}$ and $\tilde{\boldsymbol{m}}$. It is more economical to consider the Lie-generated perturbation [27, 28, 29],

$$\tilde{f}=[\tilde{g},f]+\frac{q}{m}\tilde{\boldsymbol{A}}\cdot\nabla_v f,\tag{38}$$

by solving the equation for the generating function $\tilde{g}(\boldsymbol{x},\boldsymbol{v},t)$,

$$\frac{D\tilde{g}}{Dt}=q(\tilde{\phi}-\boldsymbol{v}\cdot\tilde{\boldsymbol{A}}),\tag{39}$$

where the Lie bracket is defined by

$$\begin{aligned}[f,g]:=&\frac{1}{m}(\nabla f\cdot\nabla_v g-\nabla_v f\cdot\nabla g)\\&+\frac{q}{m^2}(\nabla_v f\times\nabla_v g)\cdot\boldsymbol{B},\end{aligned}\tag{40}$$

for any functions f,g of $(\boldsymbol{x},\boldsymbol{v},t)$. One can confirm that $\tilde{f}$ generated by (38) solves the linearized Vlasov-Maxwell equation, and moreover $\tilde{\boldsymbol{\xi}}$ and $D\tilde{\boldsymbol{\xi}}/Dt$ generated by

$$\tilde{\boldsymbol{\xi}}=[\boldsymbol{x},\tilde{g}]=\frac{1}{m}\nabla_v\tilde{g}\tag{41}$$

$$m\frac{D\tilde{\boldsymbol{\xi}}}{Dt}+q\tilde{\boldsymbol{A}}=[m\boldsymbol{v},\tilde{g}]=-\nabla\tilde{g}-\frac{q}{m}\boldsymbol{B}\times\nabla_v\tilde{g}\tag{42}$$

solve (32).

In terms of an eigenmode $(\tilde{g}, \tilde{\boldsymbol{A}})(t) = (\hat{g}, \hat{\boldsymbol{A}})e^{-i\omega t}$ + c.c., the action variable (37) is rewritten as

$$\mu = i \int \int f[\bar{\hat{g}}, \hat{g}] d^3v d^3x + 2\omega \int |\hat{\boldsymbol{A}}|^2 d^3x. \quad (43)$$

3. Spectral approach to action-angle representation

We have seen that the linearized systems for the ideal MHD equation and the Vlasov-Maxwell equation are commonly written in the canonical form (10) with a judicious choice of variable $\tilde{u}$ and its Hilbert space. Accordingly, wave action of a single eigenmode has been derived by simply applying the calculus of action variable (12) to the infinite-dimensional phase space $\tilde{u}$. However, the fluctuation $\tilde{u}(t)$ of plasma generally contains not only eigenmodes but also continuum mode. It is formally written as

$$\tilde{u}(t) = \left[\sum_n \hat{u}_n e^{-i\omega_n t} + \int_{\sigma_c} \hat{u}(\omega) e^{-i\omega t} d\omega \right] + \text{c.c.}, \quad (44)$$

in the presence of semi-simple eigenvalues (or discrete spectra) $\{\omega_n \in \mathbb{C} | n = 1, 2, 3, \dots\}$ and a continuous spectrum $\sigma_c \subset \mathbb{R}$ lying on the real axis. The naive formula (12) does not apply to continuum mode that consists of infinite number of singular (or improper) eigenmodes $\hat{u}(\omega)e^{-i\omega t}$. If a singular eigenmode was directly substituted into (12), the action variable would diverge since it is non-squire-integrable $\|\hat{u}(\omega)\|^2 = \infty$.

In order to handle multiple eigenmodes and continuum modes in a unified manner, we use the following spectral approach. By just multiplying the imaginary unit i, the linear Hamiltonian equation (10) can look like a Schrödinger equation,

$$i\partial_t \tilde{u} = \mathcal{L}\tilde{u}, \quad (45)$$

where $\mathcal{L} = i\mathcal{J}\mathcal{H}$ is a non-self-adjoint operator with respect to the inner product $\langle \bar{\circ}, \circ \rangle$ of the *complex* Hilbert space (the overbar denotes complex conjugate). We denote by $\sigma \subset \mathbb{C}$ the spectrum of $\mathcal{L}$, which is the entirety of all eigenvalues and continuous spectra.

Note that the adjoint operator is given by $\mathcal{L}^* = i\mathcal{H}\mathcal{J}$ and its spectrum is the complex conjugate $\bar{\sigma}$ of σ. The evolution equation (45) is said to possess a pseudo-Hermitian structure [30, 31], since $\mathcal{L}$ is Hermitian (i.e., self-adjoint) with respect to an indefinite inner product $\langle \bar{\circ}, \mathcal{J}\circ \rangle$.

The solution of (45) is represented by the Dunford-Taylor integral [32],

$$\tilde{u}(t) = \frac{1}{2\pi i} \oint_{\Gamma(\sigma)} (\Omega - \mathcal{L})^{-1} \tilde{u}_0 e^{-i\Omega t} d\Omega, \quad (46)$$

where $\tilde{u}_0 = \tilde{u}(0)$ denotes the initial data and $\Gamma(\sigma)$ represents a path of integration that encloses the spectrum $\sigma \subset \mathbb{C}$ counterclockwise. The operator $(\Omega - \mathcal{L})^{-1}$ is called the resolvent operator. By introducing a notation $\mathsf{U}(\Omega) = (\Omega - \mathcal{L})^{-1}\tilde{u}_0$, the Laplace transform of $\tilde{u}(t)$ corresponds to $i\mathsf{U}(\Omega)$, which analytically depends on Ω as long as Ω avoids the spectrum σ. The contour integral in (46) plays the role of the inverse Laplace transform.

We can prove that the spectrum σ is symmetric with respect to both real and imaginary axes; $\sigma = \bar{\sigma} = -\sigma = -\bar{\sigma}$, which is a common property of Hamiltonian systems [19]. [Proof: Since $\tilde{u}(t)$ must be real, $\sigma = -\bar{\sigma}$ holds. The similarity $\mathcal{J}^{-1}\mathcal{L}\mathcal{J} = \mathcal{L}^*$ implies that $\sigma = \bar{\sigma}$.]

As a preparatory for the action-angle representation of $\tilde{u}(t)$, we introduce a related Poincaré invariant (or the phase space volume enclosed by a family of solutions) [33]. First, let us decompose the spectrum into $\sigma = \sigma_+ \cup \sigma_-$ such that σ_+ (or σ_-) is inside the right (or left) half plane. According to the symmetry $\sigma_- = -\overline{\sigma_+}$, (46) is rewritten as

$$\tilde{u}(t) = \frac{1}{2\pi i} \oint_{\Gamma(\sigma_+)} (\Omega - \mathcal{L})^{-1} \tilde{u}_0 e^{-i\Omega t} d\Omega + \text{c.c.} \quad (47)$$

We generate a family of solutions by replacing the initial data $\tilde{u}_0$ by $\tilde{u}_0 e^{-i\theta_0}$,

$$\tilde{u}(t, \theta_0) = \frac{1}{2\pi i} \oint_{\Gamma(\sigma_+)} (\Omega - \mathcal{L})^{-1} \tilde{u}_0 e^{-i\Omega t - i\theta_0} d\Omega + \text{c.c.}, \quad (48)$$

which causes a uniform phase shift $(+\theta_0)$ of all eigenmodes and continuum modes. Since this family of solutions is labeled by $0 < \theta_0 \leq 2\pi$ and closed, $\tilde{u}(t, \theta_0) = \tilde{u}(t, \theta_0 + 2\pi)$, we can define a Poincaré invariant as the ensemble average over θ_0;

$$\tilde{S} := \frac{1}{4\pi} \int_0^{2\pi} \left\langle \frac{\partial \tilde{u}}{\partial \theta_0}, \mathcal{J}\tilde{u} \right\rangle d\theta_0. \quad (49)$$

It is easy to verify the invariance $\partial \tilde{S}/\partial t = 0$ by using (10), which is nothing but Liouville's theorem for the perturbation.

By substituting (47) into (49), the integration with respect to θ_0 results in

$$\tilde{S} = \mathrm{Re} \left\langle \overline{\frac{1}{2\pi i} \oint_{\Gamma'(\sigma_+)} (\Omega' - \mathcal{L})^{-1} \tilde{u}_0 e^{-i\Omega' t} d\Omega'}, \; i\mathcal{J} \frac{1}{2\pi i} \oint_{\Gamma(\sigma_+)} (\Omega - \mathcal{L})^{-1} \tilde{u}_0 e^{-i\Omega t} d\Omega \right\rangle. \quad (50)$$

By noting the identity $\mathcal{J}(\Omega - \mathcal{L})^{-1} = (\Omega - \mathcal{L}^*)^{-1}\mathcal{J}$, the Poincaré invariant is further reduced to

$$\begin{aligned}\tilde{S} =&\mathrm{Re}\frac{1}{(2\pi i)^2}\oint_{\Gamma'(\sigma_+)} d\Omega' \oint_{\Gamma(\sigma_+)} d\Omega \\ &\left\langle \overline{\tilde{u}_0}, i\mathcal{J}(\Omega'-\mathcal{L})^{-1}(\Omega-\mathcal{L})^{-1}\tilde{u}_0\right\rangle e^{i(\Omega'-\Omega)t}, \\ =&\frac{1}{2\pi i}\oint_{\Gamma(\sigma_+)} D(\Omega)d\Omega, \end{aligned} \tag{51}$$

where a function $D:\mathbb{C}\to\mathbb{C}$ is defined by

$$D(\Omega) := \left\langle \overline{\tilde{u}_0}, i\mathcal{J}(\Omega-\mathcal{L})^{-1}\tilde{u}_0\right\rangle, \tag{52}$$

and we have used a property of the resolvent operator [32]. It should be remarked that the Poincaré invariant $\tilde{S}$ is defined for general perturbations $\tilde{u}(t)$ that are not necessarily periodic in time. Namely, σ_+ may include complex eigenvalues (Im $\omega_n \neq 0$) representing growing/damping eigenmodes as well as continuous spectrum which causes the phase mixing. Note that $\tilde{S}$ defined above recovers the action variables for neutrally stable modes ($\omega_n \in \mathbb{R}$), since $\int_0^{2\pi} d\theta$ in (12) and $\int_0^{2\pi} d\theta_0$ in (49) become equivalent. Therefore, we refer to $\tilde{S}$ as the wave action of $\tilde{u}(t)$ in a wider sense.

In the last expression (51), the integration over the phase angle $0<\theta_0\le 2\pi$ is eventually converted into a contour integral in $\mathbb{C}$ enclosing the spectrum σ_+. Note that if the integral path enclosed the whole spectrum σ, it would always result in

$$\frac{1}{2\pi i}\oint_{\Gamma(\sigma)} D(\Omega)d\Omega = 0, \tag{53}$$

and mislead us into obtaining 'zero wave action'. We suggest that the correct wave action can be obtained by counting only the contribution from σ_+, i.e. the right half of the spectrum. (While we have ignored the spectrum on the imaginary axis so far, it would also turn out to be zero wave action.)

Now, we are ready to perform the *spectral decomposition* of the wave action $\tilde{S}$. Since the complex function $D(\Omega)$ is analytic except for the spectrum σ, one may analytically deform the integral path $\Gamma(\sigma_+)$ such that it consists of many closed paths that individually enclose each isolated singularity of $D(\Omega)$.

If there are semi-simple eigenvalues $\{\omega_n\in\mathbb{C}|n=1,2,3,\dots\}$, the $\mathsf{U}(\Omega)$ must have simple poles in the Ω plane,

$$\mathsf{U}(\Omega) = \frac{\hat{u}_n}{\Omega-\omega_n}+\dots. \tag{54}$$

where $\hat{u}_n$ is the projection of $\tilde{u}_0$ onto the eigenspace for ω_n. An integral path $\Gamma(\omega_n)$ surrounding ω_n gives the wave action (or the action variable) for the eigenmode,

$$\mu_n = \frac{1}{2\pi i}\oint_{\Gamma(\omega_n)} D(\Omega)d\Omega = \left\langle \overline{\tilde{u}_0}, i\mathcal{J}\hat{u}_n\right\rangle. \tag{55}$$

Strictly speaking, this is not the conventional action variable when ω_n is complex, for which the eigenmode is either exponentially growing or damping. Nevertheless, μ_n is naturally derived from the Poincaré invariant, and $\omega_n\mu_n$ indeed corresponds to the wave energy [19]. A little care needs to be paid to the fact that μ_n is complex when ω_n is complex. Due to the symmetry $\overline{\sigma_+}=\sigma_+$, an eigenvalue $\overline{\omega_n}$ also belongs to σ_+, and let $\hat{u}_{\overline{n}}$ be the corresponding projection and $\mu_{\overline{n}}$ be the 'action variable'. Using the orthogonality of the projection [32], we obtain

$$\mu_n = \left\langle \overline{\hat{u}_{\overline{n}}}, i\mathcal{J}_u\hat{u}_n\right\rangle = \overline{\left\langle \overline{\hat{u}_n}, i\mathcal{J}_u\hat{u}_{\overline{n}}\right\rangle} = \overline{\mu_{\overline{n}}}, \tag{56}$$

and hence the sum $\mu_n+\mu_{\overline{n}}$ of action variables for growing and damping modes is always real. When ω_n is a real eigenvalue, there is no distinction between ω_n and $\overline{\omega_n}$, and $\mu_n = \left\langle \overline{\hat{u}_n}, i\mathcal{J}\hat{u}_n\right\rangle \in\mathbb{R}$ agrees with the previous result (12).

As for the continuous spectrum $\sigma_c\subset\mathbb{R}$ on the real axis, the path of integration is deformed into the two paths that run parallel to σ_c at the slightly upper and lower sides;

$$\begin{aligned}&\frac{1}{2\pi i}\oint_{\Gamma(\sigma_c)} \mathsf{U}(\Omega)e^{-i\Omega t}d\Omega = \\ &\lim_{\varepsilon\to 0}\frac{i}{2\pi}\int_{\sigma_c}\left[\mathsf{U}(\omega+i\varepsilon)-\mathsf{U}(\omega-i\varepsilon)\right]e^{-i\omega t}d\omega.\end{aligned} \tag{57}$$

Hence, it is reasonable to define a singular eigenfunction for $\omega\in\sigma_c$ by

$$\hat{u}(\omega) := \frac{i}{2\pi}\left[\mathsf{U}(\omega+i0)-\mathsf{U}(\omega-i0)\right]. \tag{58}$$

This definition of $\hat{u}(\omega)$ agrees with the Fourier transform of $\tilde{u}(t)$ according to Sato's hyperfunction theory [34] (see also the Appendix of Ref. [35]). Various examples of singular eigenfunctions $\hat{u}(\omega)$ are found in literatures; see Van Kampen [36], Case [37, 38], Sedláček [39] and Tataronis [40].

The wave action for the continuous spectrum is then given as a function of ω;

$$\begin{aligned}\mu(\omega) =&\frac{i}{2\pi}\left[D(\omega+i0)-D(\omega-i0)\right] \\ =&\left\langle \overline{\tilde{u}_0}, i\mathcal{J}\hat{u}(\omega)\right\rangle.\end{aligned} \tag{59}$$

Since the initial data $\tilde{u}_0$ is usually non-singular, this $\mu(\omega)$ is well-defined.

If the spectrum σ_+ is composed of such semi-simple discrete spectrum $\{\omega_n\in\mathbb{C}:n=1,2,\dots\}$ and a real continuous spectrum $\sigma_c\subset\mathbb{R}$, the solution is represented by (44) where the complex conjugate (c.c.) stems from the other spectrum σ_-. The wave action is decomposed into the action variables,

$$\tilde{S} = \sum_n \mu_n + \int_{\sigma_c}\mu(\omega)d\omega. \tag{60}$$

Similarly, the wave energy $H^{(2)} = (1/2)\langle \partial_t \tilde{u}, \mathcal{J}\tilde{u}\rangle$ becomes

$$H^{(2)} = \frac{1}{2\pi i}\oint_{\Gamma(\sigma_+)} \Omega D(\Omega) d\Omega, \tag{61}$$

which is decomposed into

$$H^{(2)} = \sum_n \omega_n \mu_n + \int_{\sigma_c} \omega\mu(\omega) d\omega. \tag{62}$$

We remark that the derivation of the formulae, (55) and (59), becomes simpler than the previous work [19] owing to the canonical formalism. The wave actions of the vortical continuum [19] and the Alfvén and sound continua [20] have been derived from $\mu(\omega)$ in (59). Our methodology shown here is also applicable to the Vlasov-Maxwell system, and the calculation of the both wave energy and wave action will be much facilitated by standing on the present study.

4. Summary

By tracing back to the Lagrange-Hamilton theory [23, 22], we have presented the canonical form of the linearized systems for the cases of the MHD equation and the Vlasov-Maxwell equation. In the latter case, it is essential to impose the Coulomb gauge as a kinematical constraint on the fluctuation. The action-angle variable for a neutrally stable eigenmode is then easily obtained by introducing the conventional formula $\oint \boldsymbol{p}\cdot d\boldsymbol{q}$ to the corresponding phase space of the fluctuation. As for aperiodic fluctuation stemming from continuum modes and exponentially growing/damping eigenmodes, the notion of action-angle variables needs to be extended. We have naturally performed the action-angle representation for such general fluctuation by invoking the spectral technique [19] and introducing the Poincaré invariant as the ensemble average over the phase angle. The resultant expression (60) [or (62)] provides legitimate wave action (or wave energy) for any eigenmodes and continuum modes.

Historically, the wave action *density* is well-studied in the eikonal approximation [41], where the wave is locally assumed to satisfy the dispersion relation of uniform plasma. In contrast, our method corresponds to the spectral decomposition of the wave action and can deal with general fluctuation in strongly inhomogeneous media, in which the eikonal approximation is not always valid. Based on the present linear analysis, we expect better understandings of quasi-linear and weakly nonlinear behavior of plasmas.

Acknowledgments

The author is grateful to Zensho Yoshida, Yasuhide Fukumoto and Shinji Tokuda for useful comments and discussions.

A. Canonical form of the linearized Vlasov-Maxwell equation

The Hamiltonian function of the linearized system (or the wave energy) is

$$\begin{aligned} H^{(2)} =& \\ &\int d^3x \int d^3v \Bigg\{ \frac{1}{2mf}\left|\tilde{\boldsymbol{m}} - qf\left[(\tilde{\boldsymbol{\xi}}\cdot\nabla)\boldsymbol{A} + \tilde{\boldsymbol{A}}\right]\right|^2 \\ &\quad - \tilde{\boldsymbol{m}}\cdot\mathcal{D}\tilde{\boldsymbol{\xi}} + \frac{qf}{2}\tilde{\boldsymbol{\xi}}\cdot\left[(\tilde{\boldsymbol{\xi}}\cdot\nabla)\nabla(\phi - \boldsymbol{v}\cdot\boldsymbol{A})\right] \\ &\quad - qf\tilde{\boldsymbol{\xi}}\cdot\nabla(\boldsymbol{v}\cdot\tilde{\boldsymbol{A}})\Bigg\} \\ &+ \frac{1}{2}\int d^3x \left[\epsilon_0|\nabla\tilde{\phi}|^2 + \frac{1}{\epsilon_0}|\tilde{\boldsymbol{Y}}|^2 + \frac{1}{\mu_0}|\nabla\times\tilde{\boldsymbol{A}}|^2\right], \end{aligned} \tag{63}$$

where $\tilde{\phi}$ is again regarded as the solution of (29) and

$$\mathcal{D} := \boldsymbol{v}\cdot\nabla + \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\cdot\nabla_v. \tag{64}$$

The canonical equations (36) are then written as

$$\partial_t\tilde{\boldsymbol{\xi}} = -\mathcal{D}\tilde{\boldsymbol{\xi}} + \tilde{\boldsymbol{\eta}}, \tag{65}$$

$$\begin{aligned} \partial_t\tilde{\boldsymbol{m}} =& -\mathcal{D}\tilde{\boldsymbol{m}} + qf\Big[\tilde{\boldsymbol{\eta}}\times\boldsymbol{B} + (\tilde{\boldsymbol{\eta}}\cdot\nabla)\boldsymbol{A} \\ &- (\tilde{\boldsymbol{\xi}}\cdot\nabla)\nabla(\phi - \boldsymbol{v}\cdot\boldsymbol{A}) - \nabla(\tilde{\phi} - \boldsymbol{v}\cdot\tilde{\boldsymbol{A}})\Big], \end{aligned} \tag{66}$$

$$\partial_t\tilde{\boldsymbol{A}} = \frac{1}{\epsilon_0}\tilde{\boldsymbol{Y}}, \tag{67}$$

$$\begin{aligned} \partial_t\tilde{\boldsymbol{Y}} =& \mathcal{P}\left\{\int q\left[f\tilde{\boldsymbol{\eta}} - \boldsymbol{v}\nabla\cdot(f\tilde{\boldsymbol{\xi}})\right]d^3v\right\} \\ &- \frac{1}{\mu_0}\nabla\times(\nabla\times\tilde{\boldsymbol{A}}), \end{aligned} \tag{68}$$

where $\tilde{\boldsymbol{\eta}}$ should be replaced by the following expression,

$$\tilde{\boldsymbol{\eta}} := \frac{\tilde{\boldsymbol{m}}}{mf} - \frac{q}{m}\left[(\tilde{\boldsymbol{\xi}}\cdot\nabla)\boldsymbol{A} + \tilde{\boldsymbol{A}}\right]. \tag{69}$$

[1] M. D. Kruskal and C. Oberman, Phys. Fluids **1**, 275 (1958).
[2] I. Bernstein, Phys. Rev. **109**, 10 (1958), Appendix I.
[3] C. S. Gardner, Phys. Fluids **6**, 839 (1963).
[4] K. Fowler, J. Math. Phys. **4**, 559 (1963).
[5] I. B. Bernstein, E. A. Frieman, M. D. Kruskal and R. M. Kulsrud, Proc. Roy. Soc. London **A244**, 17 (1958).
[6] V. I. Arnold, *Mathematical Methods of Classical Mechanics.* (Springer, Berlin 1978).
[7] V. I. Arnold and B. A. Khesin, *Topological Methods in Hydrodynamics.* (Springer, New York 1998).
[8] D. D. Holm, J. E. Marsden, T. Ratiu and A. Weinstein, Phys. Rep. **123**, 1 (1985).
[9] M. G. Krein, Dokl. Akad. Nauk SSSR A **73**, 445 (1950).
[10] J. K. Moser, Comm. Pure Appl. Math. **11**, 81 (1958).
[11] R. MacKay, *Stability of equilibria of Hamiltonian systems*, in Nonlinear Phenomena and Chaos. S. Sarkar (ed.), 254 (1986).

[12] P. J. Morrison, Rev. Mod. Phys. **70**, 467 (1998).
[13] P. A. Sturrock, J. Appl. Phys. **31**, 2052 (1960).
[14] C. N. Lashmore-Davies, Phys. Plasmas **14**, 092101 (2007).
[15] I. V. Khalzov, A. I. Smolyakov and V. I. Ilgisonis, Phys. Plasmas **15**, 054501 (2008).
[16] R. Krechetnikov and J. E. Marsden, Rev. Mod. Phys. **79**, 519 (2007).
[17] P. J. Morrison and D. Pfirsch Phys. Fluids B, **4**, 3038 (1992).
[18] P. J. Morrison, Trans. Theory and Stat. Phys. **29**, 397 (2000).
[19] M. Hirota and Y. Fukumoto, J. Math. Phys. **49** 083101 (2008).
[20] M. Hirota and Y. Fukumoto, Phys. Plasmas **15** 122101 (2008).
[21] M. Hirota, Plasma Sci. and Tech. **11**, 409 (2009).
[22] F. E. Low, Pro. Roy. Soc. A **248** 282 (1958).
[23] W. A. Newcomb, Nucl. Fusion Suppl. Pt. 2, 451 (1962).
[24] R. L. Dewar, Phys. Fluids, **13**, 2710 (1970).
[25] J. J. Galloway and H. Kim, J. Plasma Phys. **6**, 53 (1971).
[26] E. Frieman and M. Rotenberg, Rev. Mod. Phys. **32**, 898 (1960).
[27] A. Brizard, D. R. Cook, and A. N. Kaufman, Phys. Rev. Lett. **70**, 521 (1993).
[28] J. Larsson, J. Plasma Phys. **48**, 13 (1992).
[29] P. J. Morrison and D. Pfirsch, Phys. Rev. A **40**, 3898 (1989).
[30] J. Larsson, Phys. Rev. Lett. **66**, 1466 (1991).
[31] A. Brizard, Phys. Lett. A **168**, 357 (1992).
[32] T. Kato, *Perturbation Theory for Linear Operators* (Springer-Verlag, New York, 1976).
[33] P. A. Sturrock, *Plasma Physics* (Cambridge Univ. Press, Cambridge, 1994).
[34] I. Imai, *Applied hyperfunction theory*, edited by M. Hazewinkel (Kluwer Academic Pub., Dordrecht, 1992).
[35] M. Hirota, T. Tatsuno, and Z. Yoshida, Phys. Plasmas **12**, 012107 (2005).
[36] N. G. Van Kampen, Physica **21**, 949 (1955).
[37] K. M. Case, Ann. Phys. **7**, 349 (1959).
[38] K. M. Case, Phys. Fluids **3**, 143 (1960).
[39] Z. Sedláček, J. Plasma Phys., **5**, 239 (1971).
[40] J. Tataronis and W. Grossmann, Z. Physik, **261**, 203 (1973).
[41] T. H. Stix, *The Theory of Plasma Waves.* McGraw-Hill, New York (1962).