Waveguide Modes in a Relativistic Electron Beam with Ion-Channel Guiding

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A theory for the high frequency eigenmodes of a cylindrical metallic waveguide partially filled with a relativistic electron beam and guided by an ion channel is presented. Equations that permit calculation of dispersion curves for five families of wave modes are derived. The dependence of the frequencies, and dispersion curves of azimuthally asymmetric modes on the ratio of the beam radius a to the waveguide radius R is studied in detail.

Keywords: waveguide modes, relativistic electron beam, ion-channel, dispersion relation, fluid-Maxwell equations.

1. Introduction

Ion channel may by used as an alternative guiding method to replace solenoidal or quadruple magnets in the transport of intense and relativistic electron beams. When the beam is injected into a pre-ionized channel, plasma electrons are ejected by the beam space charge and the beam electrons are then attracted to the positive-ion core. Among its many applications, ion-channel guiding of relativistic electron beam is used in advanced accelerators [1], free-electron lasers [2,3], and synchrotron x-ray radiation by betatron motion [4-6].

The dispersion characteristics of waveguide modes in a relativistic electron beam are important to be studied in many applications. The beam-frame analysis, in many occasions, is equivalent to that of the plasma column. Eigenmodes with suitable characteristics like slow wave and backward waves may be employed in devices for microwave generation. Therefore, the use of a guide that is only partially filled introduces an additional system parameter for design and control, namely, the ratio of the electron beam radius to the waveguide inner radius.

Dispersion of wave modes in a plasma column was studied in Refs. [7-10], for the completely filled and Refs. [7, 11-12], for the partially filled waveguides, respectively. Also, wave propagation in a waveguide filled with moving magnetized plasma was studied in Ref. [13].

Dispersion characteristics of waveguide modes in a relativistic electron beam with ion-channel guiding are studied in Refs. [14,15]. These analyses do not include asymmetric modes in a partially filled waveguide. The purpose of the present study is to extend the analyses of Refs. [14,15], to study the dispersion properties of asymmetric waveguide modes in a relativistic electron beam with ion-channel guiding when the guide is partially filled.

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In Sec. 2, basic equations are introduced and the dispersion relation (DR) for five families of eigenmodes are derived. In Sec. 3, the decoupled domain in which the waves separate into independent TM and TE modes is discussed. In Sec. 5, concluding remarks are made.

2. Derivation of the Dispersion Relation

A relativistic and cold electron beam with radius a passes through an ion channel with coinciding axis. Figure 1 shows the electron beam with the ion channel of radius b inside a cylindrical and metallic waveguide of radius R.

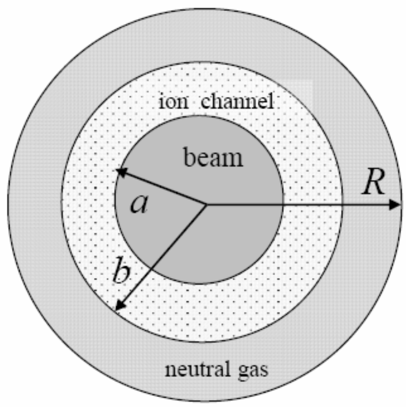

Fig.1 The cross section of the waveguide. Electron beam with radius a, ion channel with radius b, and metallic waveguide with radius R.

The equilibrium configuration has been described in details in Ref. [21], and it may be summarized as a constant axial velocity $u_{\|}$ with relativistic factor $\gamma = (1-u_{\|}^2/c^2)^{-1/2}$ plus a small amplitude transverse betatron oscillation, which is assumed nonrelativistic. The analysis is performed in the beam frame using the beam frame quantities. The investigation is restricted to the near axis region and narrow electron beam so that the small amplitude betatron motion in the equilibrium configuration can be neglected. With the self-fields of the electron beam-taken into account all electrons execute simple harmonic motion about the axis of the ion channel with betatron frequency $\omega_b = \left[(\gamma n_i - n_o)e^2/(2m\varepsilon_o)\right]^{1/2}$ in the beam frame, where n_i is the ion density in the lab frame, γn_i and n_o are the ion and beam densities in the beam frame, and e and m are the electronic charge and rest mass, respectively. In the above expression for ω_b, $\gamma n_i - n_o$ is the net charge density and the rest mass m represents the nonrelativistic oscillations of electrons in the beam frame.

Fluid-Maxwell equations will be employed to treat the perturbations and find the dispersion relation. The analysis is performed in the beam frame using the cylindrical coordinates r, θ, and z. Perturbations associated with the waves are considered, therefore, the linearized momentum equation may be written as follows [21];

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{e}{m}\mathbf{E} - \omega_b^2\ \mathbf{r}\ , \tag{1}$$

where $\mathbf{u} = d\mathbf{r}/dt + \dot{z}\hat{\mathbf{z}}$ is the perturbed velocity and $\mathbf{u}_\perp = d\mathbf{r}/dt$ is the transverse velocity. Here, the presence of ω_b^2 in Eq. (1) indicates that the betatron motion associated with the perturbation is retained. The linearized continuity equation is

$$\frac{\partial}{\partial t}n + n_o\ \nabla.\mathbf{u} = 0 \tag{2}$$

For the wave propagation, Maxwell's equations with negligible betatron motion will be written as

$$\nabla\times\mathbf{E} = i\omega\mathbf{B}, \tag{3}$$

$$\nabla\times\mathbf{B} = -\mu_0 e n_0\mathbf{u} - i\omega\varepsilon_0\mu_0\mathbf{E}, \tag{4}$$

$$\nabla\cdot\mathbf{E} = -en/\varepsilon_0, \tag{5}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{6}$$

where $e^{i(kz+l\theta-\omega t)}$ dependency is assumed. It is straight forward to express the transverse components of the fields in terms of longitudinal components [14],

$$\mathbf{B}_t = i\left\{\frac{\omega^2}{c^2}\left(\omega_p^2-\omega^2+\omega_b^2\right)-k^2\left(\omega_b^2-\omega^2\right)\right\}^{-1}\times$$
$$\left[k\left(\omega_b^2-\omega^2\right)\nabla_t B_z + \frac{\omega}{c^2}\left(\omega_p^2-\omega^2+\omega_b^2\right)\hat{\mathbf{z}}\times\nabla_t E_z\right] \tag{7}$$

$$\mathbf{E}_t = i\left\{\frac{\omega^2}{c^2}\left(\omega_p^2-\omega^2+\omega_b^2\right)-k^2\left(\omega_b^2-\omega^2\right)\right\}^{-1} \tag{8}$$
$$\times\left(\omega_b^2-\omega^2\right)\left[k\nabla_t E_z - \omega\,\hat{\mathbf{z}}\times\nabla_t B_z\right],$$

In obtaining Eqs. (7) and (8), $\mathbf{u}_\perp = d\mathbf{r}/dt = -i\omega\mathbf{r}$ is used to write the transverse and longitudinal components of the momentum Eq. (1). Equations (7) and (8) will be used later to impose the boundary conditions.

The wave equation for the longitudinal component of the electric and magnetic fields should be derived. For the electric field, take the curl of Eq. (3) and substitute for $\nabla\times\mathbf{B}$ and $\nabla\cdot\mathbf{E}$ using Eqs. (4) and (5) to obtain an equation whose z component will be the wave equation for E_z,

$$\nabla_\perp^2 E_z + \kappa_1^2\ E_z = 0, \tag{9}$$

$$\kappa_1^2 = \frac{\left(\omega^2-\omega_p^2\right)\left[k^2\left(\omega_b^2-\omega^2\right)-\frac{\omega^2}{c^2}\left(\omega_p^2+\omega_b^2-\omega^2\right)\right]}{\omega^2\left(\omega^2-\omega_p^2-\omega_b^2\right)}. \tag{10}$$

For the longitudinal component of magnetic field, take the curl of Eq. (4) and substitute for $\nabla\times E$ and $\nabla\cdot B$ using Eqs. (3) and (6) to obtain the wave equation for B_z as

$$\nabla_\perp^2 B_z + \kappa_2^2 B_z = 0, \tag{11}$$

$$\kappa_2^2 = \frac{k^2\left(\omega^2-\omega_b^2\right)+\frac{\omega^2}{c^2}\left(\omega_p^2-\omega^2+\omega_b^2\right)}{\left(\omega_b^2-\omega^2\right)}. \tag{12}$$

The wave equations inside the beam are Eqs. (9) and (11), and outside the beam we have

$$\nabla_\perp^2 E_{zv} + \kappa_v^2 E_{zv} = 0, \tag{13}$$

$$\nabla_\perp^2 B_{zv} + \kappa_v^2 B_{zv} = 0, \tag{14}$$

$$\kappa_v^2 = \frac{\omega^2}{c^2} - k^2. \tag{15}$$

The set of Eqs. (9) to (15) are the differential equations for the propagation of waves insides a waveguide partially filled with a relativistic electron beam and guided by an ion channel. The appropriate solutions are

$$E_z = c_1 J_l(\kappa_1 r)e^{i(kz+l\theta-\omega t)}, \tag{16}$$

$$E_{zv} = \left\{c_2 J_l(\kappa_v r) + c_3 N_l(\kappa_v r)\right\}e^{i(kz+l\theta-\omega t)}, \tag{17}$$

$$B_z = c_1' J_l(\kappa_2 r)e^{i(kz+l\theta-\omega t)}, \tag{18}$$

$$B_{zv} = \left\{c_2'\ J_l(\kappa_v r) + c_3' N_l(\kappa_v r)\right\}e^{i(kz+l\theta-\omega t)}, \tag{19}$$

where J_l and N_l are the Bessel functions of the first and second kind of order l.

In general, the electric and magnetic fields have all three components and the boundary conditions are the continuity of E_z, E_θ, B_z, and B_θ at $r=a$ and $r=R$, which yield six equations for six unknown constants c_1, c_2, c_3, c_1', c_2', and c_3'. Coefficients of these constants may be written as a 6×6 matrix,

$$\mathbf{C} = \begin{bmatrix}\mathbf{TM} & \mathbf{AA}\\ \mathbf{BB} & \mathbf{TE}\end{bmatrix}, \tag{20}$$

where **TM**, **TE**, **AA**, and **BB** are 3×3 matrices given by

$$\mathbf{TM} = \begin{pmatrix} J_l(\kappa_1 a) & -J_l(\kappa_v a) & -N_l(\kappa_v a)\\ 0 & J_l(\kappa_v R) & N_l(\kappa_v R)\\ i\alpha J_l'(\kappa_1 a) & i\beta J_l'(\kappa_v a) & i\beta N_l'(\kappa_v a)\end{pmatrix}, \tag{21}$$

$$\mathbf{TE} = \begin{pmatrix} J_l(\kappa_2 a) & -J_l(\kappa_v a) & -N_l(\kappa_v a) \\ 0 & -\omega|\kappa_v|J_l'(\kappa_v a) & -\omega|\kappa_v|N_l'(\kappa_v a) \\ \dfrac{\omega|\kappa_2|J_l'(\kappa_2 a)}{i\kappa_2^2} & \dfrac{i\omega|\kappa_v|J_l'(\kappa_v a)}{\kappa_v^2} & \dfrac{i\omega|\kappa_v|N_l'(\kappa_v a)}{\kappa_v^2} \end{pmatrix}, \tag{22}$$

$$\mathbf{AA} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ l\lambda J_l(\kappa_2 a) & l\delta J_l(\kappa_v a) & l\delta N_l(\kappa_v a) \end{bmatrix}, \tag{23}$$

$$\mathbf{BB} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{ilJ_l(\kappa_v R)}{Rk^{-1}} & \dfrac{ilN_l(\kappa_v R)}{Rk^{-1}} \\ l\lambda J_l(\kappa_1 a) & l\delta J_l(\kappa_v a) & l\delta N_l(\kappa_v a) \end{bmatrix}, \tag{24}$$

with

$$\alpha = (\omega^2 - \omega_p^2)\,|\kappa_1|/c^2\omega\kappa_1^2, \tag{25}$$

$$\beta = -\omega|\kappa_v|/c^2\kappa_v^2 \quad , \tag{26}$$

$$\lambda = -k/a\kappa_2^2 \quad , \tag{27}$$

$$\delta = k/a\kappa_v^2 \quad . \tag{28}$$

Here, J_l' and N_l' mean derivatives with respect to their arguments. For nontrivial solution the 6×6 matrix, in Eq. (20), which is formed by the coefficients of the fields should have zero determinant. This yields

$$\det \mathbf{C} = 0 . \tag{29}$$

The dispersion curves for the normal modes of a waveguide partially filled with a relativistic electron beam and guided by an ion channel, as ω as a function of k, may be found by numerical solution of Eq. (29) with κ_1, and κ_2 given by Eqs. (10) and (12).

3. Decoupled TM and TE Modes

The differential equations (9) and (11) for E_z and B_z are decoupled and, therefore, one of these longitudinal field components can be set independently equal to zero to obtain TM $(B_z = 0)$ and TE $(E_z = 0)$ modes. Note that in these cases all transverse components for **E** and **B** will be nonzero, which can be seen using Eqs. (7) and (8). Therefore, the boundary conditions for the TM modes do not include the continuity of E_θ. To see if this introduces any problem, note that Eq. (8) yields $E_\theta = \delta(\omega_i, \omega_p)\, lE_z$, for the TM modes. Since the coefficient $\delta(\omega_i, \omega_p)$, as given in Ref. [15], is different inside and outside the electron beam, continuity of E_z at the bean-vacuum interface does not yield the continuity of E_θ and TM modes are not possible. However, for special cases of $l = 0$ or the completely filled waveguide, in which E_z is zero on the waveguide boundary in the latter case, E_θ becomes zero and, therefore, is continuous on the surface of the beam. Similarly, B_θ is not continuous at the beam-vacuum interface for the most general case of asymmetric, i.e., $l \neq 0$, TE modes in a partially filled waveguide. Therefore, decoupled TM and TE modes are obtained for special cases of symmetric modes with $l = 0$ as well as the completely filled waveguide. The most general case of this problem is an example in which TE and TM modes are coupled, not through the wave equations, but rather through the boundary conditions.

For symmetric modes with $l = 0$, the matrix **C** in Eq. (20) is simplified as

$$\mathbf{C} = \begin{pmatrix} \mathbf{TM} & \hat{0} \\ \hat{0} & \mathbf{TE} \end{pmatrix}, \tag{30}$$

where $\hat{0}$ is the null 3×3 matrix and $l = 0$. The DR (29), in this case, reduces to

$$\det \mathbf{TM} = 0, \tag{31}$$

and

$$\det \mathbf{TE} = 0, \tag{32}$$

which yield the results in Ref. [14].

For the completely filled waveguide with a/R=1 and $l \neq 0$ the matrix **C** can not be simplified as Eq. (30) but the numerical solutions of Eqs. (31) and (32) are found to be identical with that of Eq. (29). Another important case for the decoupled TM and TE modes is at cutoff where the axial wave number k is zero. In this case, the matrix **C** in Eq. (20) reduces to Eq. (30) and the DR (29) reduces to Eqs. (31) and (32).

4. Dispersion Properties of $l = 1$ Modes

The eigenmodes of a cylindrical metallic waveguide containing a relativistic electron beam and an ion channel are characterized by wave fields that generally have both longitudinal electric and longitudinal magnetic components. These waves become purely TM or TE at three different areas as discussed in the previous section. These areas are symmetric modes with $l = 0$, a completely filled waveguide with $a/R = 1$, and cutoff frequencies at $k = 0$. We will refer to these areas as the decoupled domain. This type of waveguide has five high-frequency families of eigenmodes, namely, EH waveguide modes, HE waveguide modes, betatron modes, HE betatron modes, and space-charge modes. EH waveguide modes are the TM modes of an empty waveguide modified by the presence of the electron beam and become purely TM at the decoupled domain. The HE waveguide modes are the TE modes of an empty waveguide modified by the presence of the electron beam and become purely TE at the decoupled domain. The HE betatron (HE-Be) modes have frequencies between ω_p and ω_b and approach ω_b or ω_p whichever is larger as $k \to \infty$. These modes are purely TE at the decoupled domain. The betatron modes (Be) have frequencies, which are between the upper hybrid frequency $\omega_H = (\omega_b^2 + \omega_p^2)^{1/2}$ and the betatron frequency ω_b or plasma frequency ω_p, whichever is larger. Each mode is purely TM at the decoupled domain. The space-charge modes (SC) have frequencies that are zero at $k=0$, increase with increasing k, and approach an asymptotic limit as $k\to\infty$. This limit is ω_p or ω_b, whichever is smaller, in a completely filled guide. Each mode is purely TM at the decoupled domain.

The DR (29) is solved numerically to investigate the characteristics of some of the $l=1$ eigenmodes of a waveguide with ion-channel guided electron beam. The radius of waveguide R is normalized by the factor ω_p/c and has the value $R\omega_p/c=1.2$, which corresponds to $R=2$ mm for $\omega_p=1.8\times10^{11}$ rad/s. The electron density, in the beam frame, is $n_0=1.03\times10^{13}cm^{-3}$ and the Lorentz factor is taken to be $\gamma=6$.

Dispersion curves for the space-charge SC_{11} mode and HE betatron HE-Be_{11} mode are shown in Fig. 2, for the completely filled waveguide $a/R=1$ and partially filled waveguides with $a/R=0.7$ and 0.3. At a given wave number, space-charge waves have higher frequency for the completely filled waveguide compared to when the beam radius is less than the guide radius. The frequencies of HE-Be_{11} mode, on the other hand, become higher as the filling factor a/R is reduced.

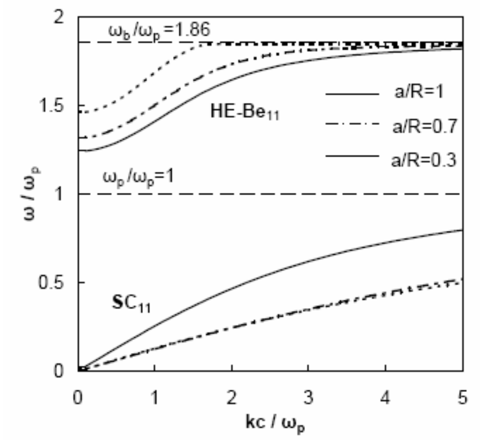

Fig.2 Dispersion curves for SC_{11} and *HE-*Be_{11} modes.

Dispersion curves for the waveguide modes EH_{11} and HE_{11} are shown in Fig. 3(A). For both modes, dispersion curves for partially filled waveguides lie below the curve for the completely filled guide. These curves approach those of an empty waveguide as a/R is reduced to zero. Dispersion curves for the betatron Be_{11} and Be_{12} modes are shown is Fig. 3(B). Be_{11} mode behaves differently for $a/R=0.3$ due to the interaction. The cutoff frequencies, except for the Be_{11} mode with $a/R=0.3$, are not noticeably less than ω_H, and the frequency of this mode, at a given wave number, gets closer to ω_H as the filling factor a/R becomes less.

The dependence of frequencies of $l=1$ modes on the beam radius a to waveguide radius R is illustrated in

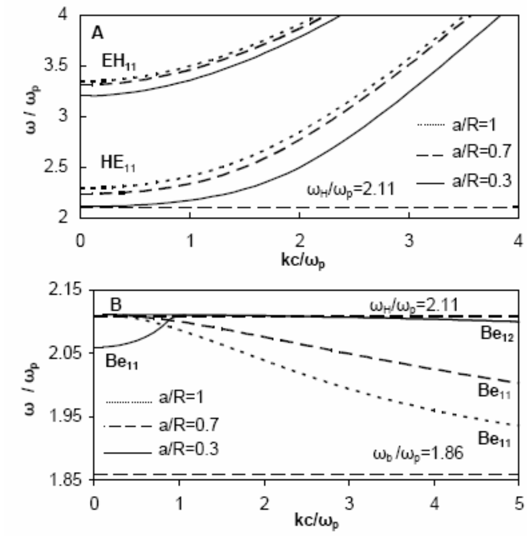

Fig.3 Dispersion curves for EH_{11} and HE_{11} modes (A) and for Be_{11} and Be_{12} modes (B).

Fig. 4 for $\omega_b/\omega_p=1.86$ and $kc/\omega_p=3$. $a/R=0$ and 1 correspond to an empty waveguide and a completely filled waveguide, respectively. For $a/R=0$, the frequencies approach the zero frequency for the space-charge mode and the upper-hybrid frequency $\omega_H/\omega_p=2.11$, with diminishing amplitude, for the betatron mode. In an empty waveguide, the betatron and space-charge modes do not exist. Variation of the frequency with a/R, for the space-charge modes, is considerably larger than that of the betatron modes. The frequency for the space-charge modes as well as the EH and HE modes increases with increasing a/R while it decreases for the betatron and HE-betatron modes.

5. Conclusion

In this article, dispersion characteristics of asymmetric $l=1$ eigenmodes in a relativistic electron beam with ion-channel guiding are studied. The analysis is performed in the beam frame with self-fields taken into account. The steady state is in ion-focused regime and the analysis is limited to near axis waves with negligible betatron motion. Numerical study of the dispersion curves of space-charge modes reveals a strong dependence on the ratio of beam radius a to waveguide radius R. In contrast, dispersion curves and cutoff frequencies for the EH and HE waveguide modes, betatron modes, and HE-betatron modes are only moderately dependent on the radius ratio.

One of the applications of relativistic electron beams is in the generation high power (around 40 MW) and coherent electromagnetic radiation, with free-electron laser being an example. The intrinsic efficiency in this device

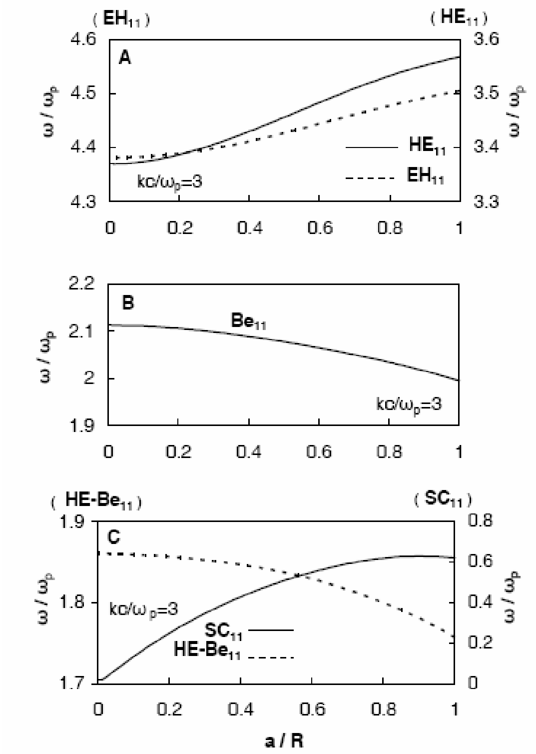

Fig.4 Normalized frequency as a function of electron beam radius a to waveguide radius R for waveguide modes EH_{11} and HE_{11} (A), betatron mode Be_{11} (B), space-charge mode SC_{11} and $HE\text{-}Be_{11}$ mode (C).

is around a few percent. However, there are efficiency enhancement mechanisms by which the efficiency can reach as high as 25 percent.

6. Acknowledgment

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7. References

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