CONJUGATE VARIABLE METHOD FOR APPLYING TO MHD STABILITY ANALYSIS

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The conjugate variable method is discussed for the eigenvalue or boundary-value problems of the ordinary differential equations that appear in the magnetohydrodynamic stability analysis. The method is used to the reduced MHD equations to derive a canonical 1-form of them. The 1-form thus obtained provides the initial step for the application of the Hamilton-Lie perturbation theory.

Keywords: conjugate variable, Hamiltonian dynamics, MHD stability analysis, reduced MHD equations, Hamilton-Lie perturbation method

1. Introduction

The conjugate variable method is well-known in the path-integral formalism of classical statistical dynamics[1]. The method endows a system of ordinary differential equations with the Hamiltonian structure by doubling the unknown variables. Recently, the author studied the conjugate variable method to demonstrate that it can be effectively used to apply the canonical Hamilton-Lie perturbation theory to a system that does not have the Hamiltonian structure[2].

The Hamilton-Lie perturbation method has been mainly used in the problems of classical dynamics (initial-value problems for ordinary differential equations), and has not been applied to the problems such as eigenvalue problems in MHD (MagnetoHydroDynamic) stability theory. One reason will be the non-Hamiltonian structure of the MHD problems. The present paper applies the conjugate variable method to the reduced MHD equations and derive the canonical 1-form of them. It provides the first step for further investigation of the reduced MHD equations from a view point of the Hamilton-Lie perturbation theory. In Section 2, the conjugate variable method is illustrated by applying it to a model equation in the MHD stability analysis. In Section 3, the canonical 1-form of the reduced MHD equations is derived by using the conjugate variable method, and summary is given in Sec.4.

2. Eigenvalue Problem in MHD Stability Analysis

Let us consider

$$\frac{d}{dx}\left(f(x,\lambda)\frac{dy}{dx}\right)+g(x,\lambda)y(x)=0, \tag{1}$$

where $f(x,\lambda)>0$ is implied in the domain $x\in[a,b]$. Instead of initial-value problems, we are interested in boundary-value or eigenvalue problems for Eq.(1), and therefore we want to determine the parameter λ so that the solution $y(x)$ satisfies the imposed boundary condition. An example in the MHD theory is the model equation of ballooning modes[3], where x is the so called extended poloidal angle of $-\infty<x<\infty$. Note that Eq.(1) has the Lagrangian

$$\mathcal{L}=\frac{1}{2}\int_a^b\left[f(x,\lambda)(\frac{dy}{dx})^2-g(x,\lambda)y^2\right]dx, \tag{2}$$

and therefore, it is easy to construct a 1-form that generates Eq.(1). One is the canonical 1-form given by

$$\gamma=pdy-h(y,p,x)dx, \tag{3}$$

where the Hamiltonian is

$$h(y,p,x)=\frac{p^2}{2f(x,\lambda)}+\frac{g(x,\lambda)}{2}y^2. \tag{4}$$

Another is the non-canonical 1-form[4] given by

$$\gamma=f(x,\lambda)udy-H(y,u,x)dx, \tag{5}$$

where

$$H(y,u,x)=\frac{f(x,\lambda)}{2}u^2+\frac{g(x,\lambda)}{2}y^2. \tag{6}$$

It is easy to identify each of the Hamilton equations generated from Eqs.(3) and (5) as Eq.(1). Equation (4) can be interpreted as the Hamiltonian of a harmonic oscillator whose mass and spring constant vary with the *time* x. We will not meet any difficulty of applying the Hamilton-Lie perturbation method to Eq.(1) since we have the corresponding 1-form, Eq.(3) or Eq.(5).

However, 1-forms such as Eqs.(3) and (5) assume the existence of Lagrangian, which is a too strong limitation to apply more general problems. The assumption of Lagrangian becomes unnecessary when the conjugate variable method is used to differential equations. Let us consider the *normal* form of the differential equation (1)

$$\frac{dy}{dx}=\frac{p}{f(x,\lambda)}, \tag{7}$$

$$\frac{dp}{dy}=-g(x,\lambda)y. \tag{8}$$

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Next, by introducing the conjugate variables α (*respectively* β) for y (*respectively* p), we can make the canonical 1-form

$$\gamma = \alpha dy + \beta dp - H dx, \tag{9}$$

and

$$H = \frac{\alpha p}{f(x,\lambda)} - g(x,\lambda)\beta y. \tag{10}$$

Also, by transforming the variables

$$\alpha = f(x,\lambda)A, \quad \beta = B, \tag{11}$$

we obtain

$$\gamma = f(x,\lambda)A dy + B dp - H dx, \tag{12}$$

and

$$H = Ap - g(x,\lambda)By. \tag{13}$$

Equation (12) is the non-canonical 1-form with conjugate variables.

Here let us note that the conjugate variable method can be easily applied to non-linear differential equations as long as they are expressed in a normal form.

3. Example in the Reduced MHD equations

Next, let us consider, as an example that does not have a Lagrangian, the low beta reduced MHD model for a cylindrical plasma[3]. In this model, the magnetic field $\mathbf{B}$ and the velocity of a plasma $\mathbf{v}$ are expressed as

$$\mathbf{B} = \mathbf{e}_z \times \nabla(-i)\psi(\mathbf{x},t) + B_z \mathbf{e}_z, \tag{14}$$

and

$$\mathbf{v} = \mathbf{e}_z \times \nabla\phi(\mathbf{x},t), \tag{15}$$

where the axial magnetic field, B_z, is constant in time and uniform in space.

Let $\psi_0(r)$ be the poloidal flux function in an equilibrium state. Then the safety factor $q(r)$ is given by

$$q(r) = \frac{r}{R_0}\frac{B_z}{d\psi_0/dr}, \tag{16}$$

with using the cylindrical coordinate system (r,θ,z); $2\pi R_0$ is the length of a plasma column. Also, the current density in the axial direction is given by

$$j_z(r) = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi_0}{dr}\right). \tag{17}$$

Assuming that $\psi, \phi \propto \exp(\lambda t + im\theta - ikz)$, the reduced MHD model yields linearized equations for ψ and ϕ as

$$\lambda\triangle_\perp\phi = K_\parallel(r)(\triangle_\perp\psi) - \frac{dj_z}{dr}\frac{m}{r}\psi, \tag{18}$$

and

$$\lambda\psi = -K_\parallel(r)\phi + \eta\triangle_\perp\psi \tag{19}$$

Here $\eta > 0$ is the plasma resistivity. The operator $\triangle_\perp$ is defined by

$$\triangle_\perp\psi = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) - \left(\frac{m}{r}\right)^2\psi, \tag{20}$$

and the parallel wave number $K_\parallel(r)$ by

$$K_\parallel(r) = \frac{B_z}{R_0}\left(\frac{m}{q(r)} - n\right), \quad n = kR_0. \tag{21}$$

When $\eta = 0$, Eqs.(18) and (19) reduce to the ideal MHD equation with respect to ϕ, and its Lagrangian is well-known. When $\eta \neq 0$, such a Lagrangian does not exit, then the conjugate variable method will prove to be useful in constructing 1-form. To this end, we rewrite Eqs.(18) and (19) into a normal form. We first introduce the generalized momenta, Φ and Ψ, by

$$\Phi = r\frac{d\phi}{dr}, \quad \Psi = r\frac{d\psi}{dr}. \tag{22}$$

Then Eqs.(18) and (19) become first order differential equations for Φ and Ψ, which are given by

$$\begin{aligned}\frac{d\Phi}{dr} &= \left[rK_\parallel^2\frac{1}{\eta\lambda} + \frac{m^2}{r}\right]\phi \\ &+ \left[r\frac{K_\parallel}{\eta} - \frac{m}{\lambda}\frac{dj_z}{dr}\right]\psi,\end{aligned} \tag{23}$$

and

$$\frac{d\Psi}{dr} = r\frac{K_\parallel}{\eta}\phi + \left[\frac{\lambda}{\eta}r + \frac{m^2}{r}\right]\psi. \tag{24}$$

Now it is easy to construct the 1-form for the reduced MHD equations, Eqs.(18) and (19). Introducing conjugate variables P, Q, X and Y for ϕ, ψ, Φ and Ψ, respectively, the 1-form is expressed as

$$\gamma = Pd\phi + Qd\psi + Xd\Phi + Yd\Psi - hdr, \tag{25}$$

and the Hamiltonian h is given by

$$\begin{aligned} h &= \frac{1}{r}(P\Phi + Q\Psi) \\ &+ h_{17}\phi X + h_{27}\psi X \\ &+ h_{18}\phi Y + h_{28}\phi Y, \end{aligned} \tag{26}$$

where

$$h_{17} = rK_\parallel^2\frac{1}{\eta\lambda} + \frac{m^2}{r}, \tag{27}$$

$$h_{27} = r\frac{K_\parallel}{\eta} - \frac{m}{\lambda}\frac{dj_z}{dr}, \tag{28}$$

$$h_{18} = r\frac{K_\parallel}{\eta}, \tag{29}$$

and

$$h_{28} = \frac{\lambda}{\eta}r + \frac{m^2}{r}. \tag{30}$$

4. Summary

The canonical 1-form of the reduced MHD equations has been derived by the conjugate variable method. This method is effective for wide problems in MHD stability analysis such as finite beta plasma in tokamaks as long as the MHD equations are expressed as ordinary differential equations on the poloidal Fourier harmonics of the mode.

If once we make the 1-form for an eigenvalue problems, we can apply the Hamilton-Lie perturbation analysis of the problem. Further investigation will be reported in the near future.

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