Characteristics of SVD in ST Plasma Shape Reproduction Method Based on CCS

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Least square method has been used to solve the inverse problem from flux loop measurement to the boundary values on the CCS (Cauchy Condition Surface). When the CCS method is applied to real experimental data, noise superposition is inevitable. By introduction of SVD (Singular Value Decomposition) and truncation of the least SV components, we can expect the shape reproduction robustness against the noise. The truncation simulation of the less SV component shows shape reproduction error less than no truncation, when the measurement error exceeds a certain value.

Keywords: SVD, plasma shape reproduction, CCS, least square method, flux loop, magnetic probe, noise superposition.

1. Introduction

CCS (Cauchy Condition Surface) method is a numerical approach to reproduce plasma shape, which has good precision in conventional tokamak [1]. In order to apply it in plasma shape reproduction of ST (Spherical Tokamak), the calculation precision of the CCS method in ST has been analyzed [2]. The precision was confirmed also in ST, and rather difficulties in calculation of ST equilibrium configuration with low aspect ratio were found. Least square method has been used to solve the inverse problem from flux loop measurement to the boundary values on the CCS. When the CCS method is applied to real experimental data, noise superposition is inevitable. By introduction of SVD (Singular Value Decomposition) and truncation of the least SV components, we can expect the shape reconstruction robustness against the noise, since the measurement error affects the solution error in proportion to (not the square of) the condition number, which is defined as maximum SV divided by minimum SV. The truncation simulation of the least SV component shows shape reproduction error less than no truncation, when the measurement error exceeds a certain value.

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2. Shape Reconstruction by CCS Method

The Cauchy-Condition Surface method is a kind of exact numerical method, which is based on the boundary integral equation. The Cauchy-Condition surface is defined as a hypothetical plasma surface, where both the Dirichlet (ϕ) and Neumann (B_t) conditions are unknown. This surface is located inside the real plasma region. It is assumed that CCS encloses all the plasmas and there are no plasmas outside the CCS [1].

According to the static Maxwell's equation, three types of boundary integral equations can be given by using the magnetic sensor signals and poloidal coil current data [1].

$$8\pi^2\phi(\boldsymbol{x}_f) = -\int_{\partial\Omega}\left[G(\boldsymbol{x}_f,\boldsymbol{z})\mathrm{grad}\phi(\boldsymbol{z}) - \phi(\boldsymbol{z})\mathrm{grad}G(\boldsymbol{x}_f,\boldsymbol{z})\right]\frac{\mathrm{d}\boldsymbol{S}(\boldsymbol{z})}{R_z^2} + \int_{\Omega}\mu_0\boldsymbol{J}_C(\boldsymbol{y}).G(\boldsymbol{x}_f,\boldsymbol{y})\frac{\mathrm{d}V(\boldsymbol{y})}{R_y} \quad (1)$$

$$Bt(x_B) = -\boldsymbol{n}.\frac{\mathrm{grad}\phi(\boldsymbol{x}_B)}{R_B} \quad (2)$$

$$4\pi^2\phi(\boldsymbol{x}) = -\int_{\partial\Omega}\left[G(\boldsymbol{x},\boldsymbol{z})grad\phi(\boldsymbol{z}) - \phi(\boldsymbol{z})gradG(\boldsymbol{x},\boldsymbol{z})\right]\frac{\mathrm{d}\boldsymbol{S}(\boldsymbol{z})}{R_z^2} + \int_{\Omega}\mu_0\boldsymbol{J}_C(\boldsymbol{y}).G(\boldsymbol{x},y)\frac{\mathrm{d}V(\boldsymbol{y})}{R_y} \quad (3)$$

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Where $\phi(\boldsymbol{x}_f)$ is poloidal flux function at flux loop position, $\phi(\boldsymbol{z})$ is poloidal flux function on CCS, G is Green's function (a poloidal flux function for toroidal filament current in axisymmetric geometry) and $\boldsymbol{J}_C(\boldsymbol{y})$ is current density in eq. (1). $Bt(\boldsymbol{x}_B)$ is magnetic field at magnetic probe position and $\boldsymbol{n}$ is normal vector perpendicular to the magnetic probe in eq. (2). $\phi(\boldsymbol{x})$ is poloidal flux function on CCS and $\phi(\boldsymbol{z})$ is also poloidal flux function on CCS in eq. (3).

The discretized formulas for flux loops, magnetic probes and CCS are as follows [2].

$$\phi(x_f)=\sum_{i=1}^{M}W_{F1}(x_f,z_i)\phi(z_i)+\sum_{i=1}^{M}W_{B1}(x_f,z_i)\mathrm{B}t(z_i)+W_{C1}(x_f)I_{PF} \quad \text{(1a)}$$

$$Bt(x_B)=\sum_{i=1}^{M}W_{F2}(x_B,z_i)\phi(z_i)+\sum_{i=1}^{M}W_{B2}(x_B,z_i)\mathrm{B}t(z_i)+W_{C2}(x_B)I_{PF} \quad \text{(2a)}$$

$$\frac{1}{2}\phi(x)=\sum_{i=1}^{M}W_{F3}(x,z_i)\phi(z_i)+\sum_{i=1}^{M}W_{B3}(x,z_i)\mathrm{B}t(z_i)+W_{C3}(x)I_{PF} \quad \text{(3a)}$$

Where, M is the number of discretized points along CCS. $\phi(x_f)$ is the flux loop measurement. $Bt(x_B)$ is the magnetic probe measurement. $\phi(z_i)$ and $\mathrm{B}t(z_i)$ are the flux and B_t value (tangential component), repectively, of discretized points on CCS. I_{PF} is the poloidal field coil current included in the calculation region. $W_{F1}(x_f,z_i)$, $W_{B1}(x_f,z_i)$, $W_{C1}(x_f)$, $W_{F2}(x_B,z_i)$, $W_{B2}(x_B,z_i)$, $W_{C2}(x_B)$, $W_{F3}(x,z_i)$, $W_{B3}(x,z_i)$, $W_{C3}(x)$ are coefficient matrices which can be calculated beforehand. In eqs. (1a), (2a) and (3a), $\phi(x_f)$, $Bt(x_B)$ and I_{PF} are known (measured) values, and $\phi(z_i)$ and $\mathrm{B}t(z_i)$ are unknown values. Though the number of the unknown values is $2M$, the final number of the unknown values is M, since $\phi(z_i)$ and $\mathrm{B}t(z_i)$ are related through eq. (3a).

Equations (1a), (2a) and (3a) are coupled and the observation equation can be expressed in matrix form, and then $\phi(z_i)$ and $\mathrm{B}t(z_i)$ at several discretized points along CCS can be evaluated by using the least square method.

Then the flux distribution can be calculated using equation (4), and the outmost magnetic flux surface or plasma shape can be found by plotting the contour.

$$\phi(x)=\sum_{i=1}^{M}W_{F4}(x,z_i)\phi(z_i)+\sum_{i=1}^{M}W_{B4}(x,z_i)\mathrm{B}t(z_i)+W_{C4}(x)I_{PF} \quad \text{(4)}$$

Where, $\phi(x)$ is the flux value at arbitrary point, and $W_{F4}(x,z_i)$, $W_{B4}(x,z_i)$ and $W_{C4}(x)$ are coefficient matrices.

3. Least Square Method and SVD

The observation equation is $Ax=b$, where A is coefficient matrix (not square), x is unknown value (flux and Bt value) on CCS and b is known value measured by flux loop and/or magnetic probe. In case of least square method, $(Ax-b)^t(Ax-b)$ is minimized and $(A^tA)x=(A^tb)$ is solved as $x=(A^tA)^{-1}(A^tb)$, where superscript t denotes transpose. Therefore the measurement error affects the solution error in proportion to square of the condition number, which is defined as maximum SV divided by minimum SV. In case of SVD, however, A is decomposed as (UWV^t) and $(UWV^t)x=b$ is solved as $x=(VW^{-1}U^t)b$. Therefore the measurement error affects the solution error in proportion to the condition number.

In case of CPD (Compact PWI experimental Device), for example, SV is shown in Fig. 1, the singular vector U and V are shown in Fig. 2 with the measured flux value. Figure 3 shows the error and the contribution fraction, which is defined as the accumulated SV squared. The error decreases by accumulation of even mode, since the plasma shape is symmetrical with respect to the equatorial plane in this case. The plasma shape approaches the true shape with accumulation of the modes as shown in Fig. 4. If the least SV component is neglected, the detailed shape is not reconstructed, but the measurement error may not affect the reconstructed shape error, since the condition number decreases.

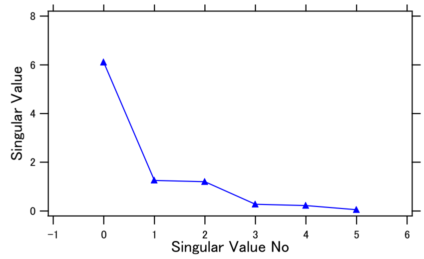

Fig.1. Singular values in descendeing order.

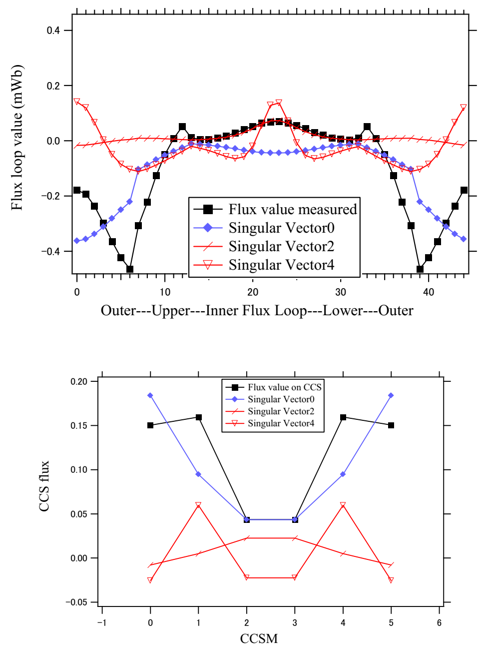

Fig.2. (a) Eigen vectors U and flux values at flux loop, (b) Eigen vectors V and flux values on CCS in anti-clockwise beginning at the outboard.

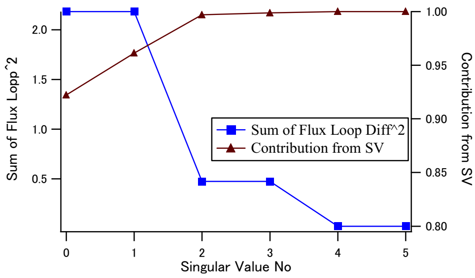

Fig.3. Error and contribution fraction in decreasing order of SV.

4. Noise Effect in SVD

The observation equation is $Ax=b$, where A is coefficient matrix (n, m) $(n>m)$, x is unknown vector $(m, 1)$ on CCS and b is known vector $(n, 1)$ measured by flux loop and/or magnetic probe. In case of least square method, normal equation $(A^t A)x=(A^t b)$ is calculated and it is solved as $x=(A^t A)^{-1}(A^t b)$. Therefore the effect of rounding errors $\|\Delta x\|/\|x\| \le \kappa^2 \|\Delta b\|/\|b\|$, where the condition number is defined as the ratio of singular values $\kappa \equiv \mu_{\max}(A)/\mu_{\min}(A)$. When the rank of A, however, is full rank $r=\mathrm{rank}(A)=m$ and A is decomposed as $A=UWV^t$, (where U is matrix (n, m) of singular vectors, W is diagonal matrix (m, m) of singular values, V is matrix (m, m) of singular vectors,) the above solution $x=(A^t A)^{-1}(A^t b)$ is the same as the solution $x=(VW^{-1}U^t)b$. Therefore the effect of rounding errors is the same, when the rounding errors are negligibly small in matrix calculation.

Figure 5 shows noise dependence of maximum flux loop difference (from the true value calculated by equilibrium code [3]) divided by the flux loop value, where measure noise is simulated by random noise. When less SV is truncated, noise effect decreases, though the detailed information is not transferred from flux loop value to the values on CCS. Therefore the error increases relatively slowly in the large noise region and the

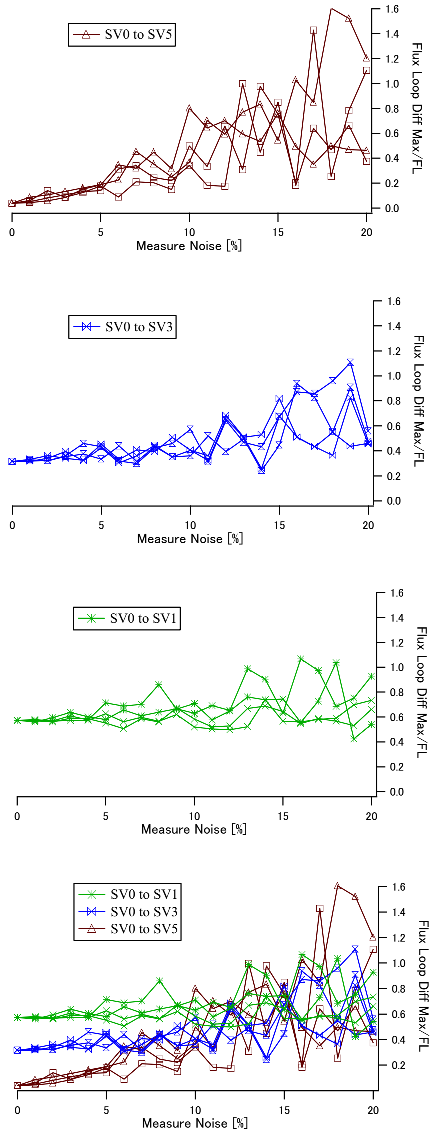

Fig. 5. Noise dependence of maximum flux loop difference divided by the flux loop value in case of only first SV, 1st-3rd SV, 1st-5th SV and all dependencies.

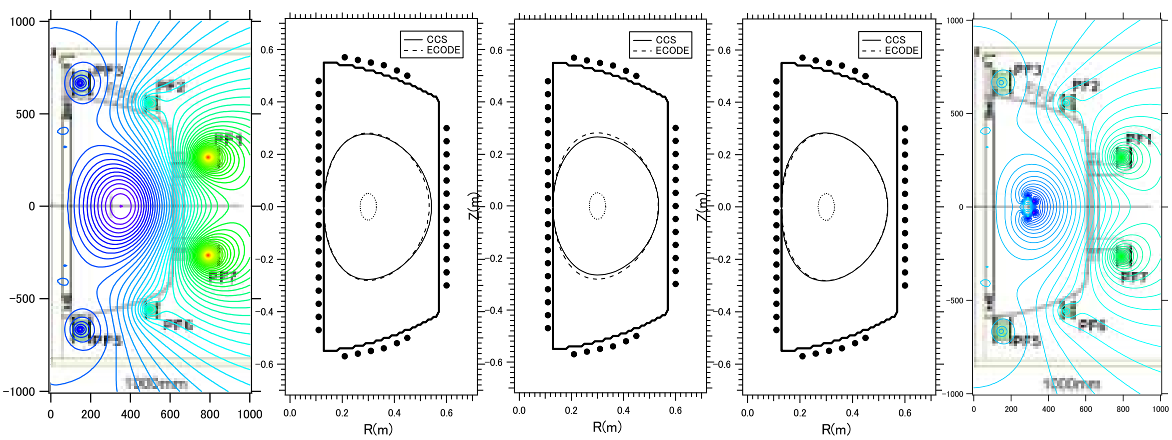

Fig.4. Original plasma shape, reproduced shapes of only first SV, 1st-3rd SV, 1st-5th SV and all SV. In the 2 to 5th figures, solid line is poloidal limiter and solid circles are flux loops. Small dotted circle inside the plasma boundary is CCS.

deviation is relatively small, though the error increases in the small noise region. The truncation simulation of the less SV component shows shape reproduction error less than no truncation, when the measurement error exceeds a certain value (15 % in this case). In this configuration, the ST plasma is symmetrical with respect to the equatorial plane and the number of free parameters on CCS is effectively half of 6. When the plasma shape is not symmetrical vertically or the degree of freedom is increased, the certain value for inversion may decrease.

5. Summary

CCS (Cauchy-Condition Surface) method is a numerical approach to reproduce plasma shape, which has good precision in conventional tokamak. Least square method has been used to solve the inverse problem from flux loop measurement to the boundary values on the CCS. When the CCS method is applied to real experimental data, noise superposition is inevitable. By introduction of SVD (Singular Value Decomposition) and truncation of the least SV components, we could show the shape reconstruction robustness against the noise.

When less SV is truncated, the error increases relatively slowly in the large noise region and the deviation is relatively small, though the error increases in the small noise region. The truncation simulation of the less SV component shows shape reproduction error less than no truncation, when the measurement error exceeds a certain value. In this configuration, the ST plasma is symmetrical with respect to the equatorial plane and the number of free parameters on CCS is effectively half of 6. When the plasma shape is not symmetrical vertically or the degree of freedom is increased, the certain value for inversion may decrease.

6. References

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