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APPROPRIATENESS EXAMINATION OF THE MODEL FUNCTIONS FOR THE EQUILIBRIUM RECONSTRUCTION ANALYSIS

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We have developed a reconstruction code for axi-symmetric plasma using the finite element method and the nonlinear least-squares method using experimentally measured data; this code can be used for the analysis of stability and positional control in magnetically confined plasma. We examine whether the model function of the magnetic surface function can adequately reflect the structure using the measurement data and also examine the measurement accuracy by means of a $\chi 2$ test and by Akaike's information criterion. Furthermore, the large number of calculations required using this code are performed at a sufficiently high speed by using semi-infinite elements and parallel computation using the parallel iterative solver of GeoFEM.

Keywords: MHD equilibrium, axi-symmetric plasma, Grad-Shafranov equation, Equilibrium reconstruction, Magnetic surface function, Least-squares method, Parallel computing

1. Introduction

It is necessary to determine the two-dimensional distribution of magnetic flux and the pressure for the analysis of magnetohydrodynamic (MHD) stability and transport. This distribution is synthetically derived from the data of several experimentally measured parameters such as magnetic flux densities, plasma temperature, density inside and outside the plasma, toroidal plasma current, etc.; it enables a more accurate analysis and allows a comparison with the relaxation theory of magnetically confined plasma[1].

The equilibrium configuration of an axi-symmetric plasma described by the Grad-Shafranov equation is calculated by assuming two surface model functions $I(\Psi) = rBt$ and $P(\Psi)$, where r denotes the radial variable of the cylindrical coordinate (r, θ, z); Bt, the toroidal magnetic flux density; and P, the kinetic pressure. Further, using the nonlinear least-squares method (N-LSM) and FEM equilibrium code, we can obtain the functions $I(\Psi)$ and $P(\Psi)$ and the equilibrium by minimizing a residual. For this calculation, there exists a technique that can be used to determine the magnetic surface ψ using Green's function method[2]. This technique does not require too much computing time. However, although our FEM code requires more time to determine ψ, we can calculate the magnetic surface more accurately and also determine the boundary conditions.

Next, the model functions $I(\Psi)$ and $P(\Psi)$ are used to express more various continuous function, and to obtain the property that S has a single minimum point even if there exists a measurement error. We examine the statistical significance of the obtained model functions by means of a χ^2 test and by Akaike's information criterion.

In this code, a large number of equilibrium calculations are performed to obtain a converged solution of the N-LSM. Therefore, the processes used to calculate the Jacobian matrix and the FEM analysis are parallelized using MPI and the GeoFEM[3] routine. We also examine how the use of this approach improves the calculation times.

2. Equilibrium Reconstruction Analysis

Equilibrium reconstruction code consists of the equilibrium analysis code solved Grad-Shafranov equation renewing the parameters of model function one after another and Marqurt code of non linear least-squares method minimized the residual between the experimental measurement data and calculated ones. Figure1 is the flowchart of equilibrium reconstruction code.

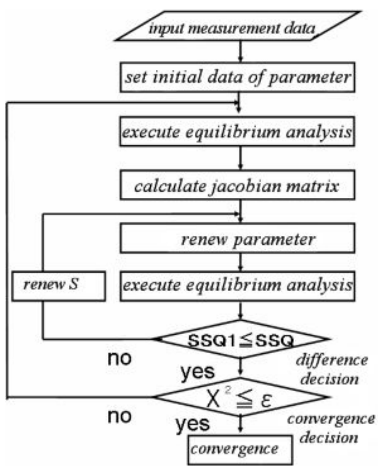

Fig1. Flow Chart of Equilibrium reconstruction

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2.1 Equilibrium analysis code EAFP

Eq.(1) is the Grad-Shafranov equation in the cylindrical coordinates showed in Fig.2. Ψ=const shows the magnetic surface and the functions I(Ψ) and P(Ψ) are depended only on the magnetic surface function Ψ. Therefore, the problem to find the magnetic distribution is considered to be same as that to find unknown functions I(Ψ) and P(Ψ).

$$\nabla \cdot \frac{1}{r^2} \nabla \psi = -\frac{\mu_0}{r} J_\theta$$

$$J_\theta = \begin{cases} r\dfrac{\partial P(\psi)}{\partial \psi} + \dfrac{I(\psi)}{\mu_0 r}\dfrac{\partial I(\psi)}{\partial \psi} & \textit{in plasma} \\ J_{\phi c} & \textit{in vacuum} \end{cases} \tag{1}$$

In our study, we used EAFP(Toroidal Equilibrium Analysis of an Axi-symmetric Free-Boundary Plasma) as equilibrium analysis code[4]. EAFP is a finite element method code which solved Eq.(1) as a free boundary problem on the condition of the magnetic surface functions I(Ψ) and P(Ψ), the shape of vacuum vessel and the conducting shell.

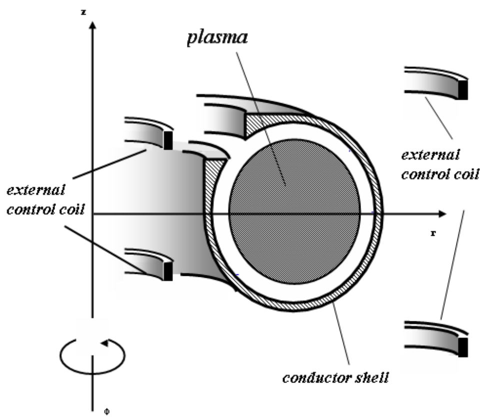

Fig.2 Geometry of an axi-symmetric plasma

2.2 Least-Squares Method

Using parameters x_1, x_2, ..,x_{n+m}, I(Ψ) and P(Ψ) are expressed as follow:

$$I(\psi) = g(\psi, x_1, x_2, \cdots, x_m) \tag{2}$$

$$P(\psi) = h(\psi, x_{m+1}, x_{m+2}, \cdots, x_{m+n}) \tag{3}$$

EAFP calculated physical values $\{y_i\}$, such as magnetic flux density and kinetic pressure in figure3, temperature, electron density at several point of plasma, are the function of the parameter. Therefore, we renew the values to minimize the residue between the measurement data(test data) y_i* with the value of equilibrium analysis y_i given by

$$SSQ = \sum_{j=1}^{n} w_j \left(y_i^* - y_i\right)^2 \tag{4}$$

, and can get the solution of Grad-Shafranov equation fitting measurement data and so we find the practical magnetic surface function and distribution. For a non-linear least square method, we use the Marquardt algorithm[5] that has the merit of steepest descent method and Gauss-Newton method.

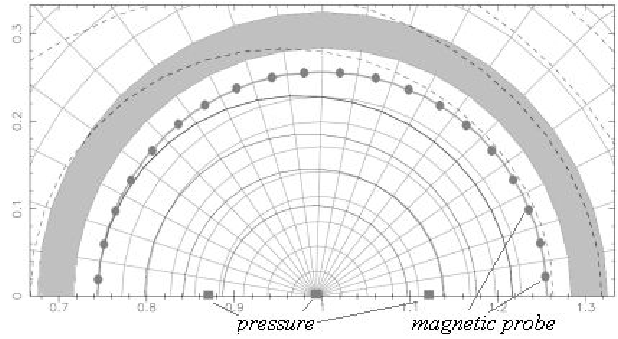

Fig.3 Positions of Measurement Data

2.3 Parallel calculation of the Equilibrium calculation

We use the solver of a simultaneous linear equation of GeoFEM[6] for speeding up equilibrium analysis, where GeoFEM is the parallel finite element analysis system developed with development of earth simulator. The function of GeoFEM solver supports communication table and it is possible for this to develop parallel finite element method code.

3. Appropriateness examination

In our study, we analyze an equilibrium using the geometrical shape of TPE-2M RFP machine in the National Institute of Advanced Industrial Science and Technology in Japan. Figure4 shows a typical distribution of Ψ. And to examine the effectiveness of our reconstruction analysis code and the model functions, as the test data, we use $y_i^*=y_i + \delta y_i$, where y_i^*'s are EAFP analyzed values using the functions as

$$P^*(\psi) = 1.5\beta_0 \tilde{\psi}\left(1 - \frac{\tilde{\psi}^2}{3}\right)$$
$$I^*(\psi) = 1 + \lambda\left(\tilde{\psi} - 1 + \frac{(1-\tilde{\psi})^{\alpha+1}}{\alpha+1}\right) \tag{5}$$

(β_0=0.1, λ=1.2, α=10) and δy_i's are normal random numbers with variance σ. $\tilde{\psi}$ is the value that standardize by Ψ_b and Ψ_c (Ψ_b and Ψ_c is Ψ in plasma surface and magnetic axis). We set a polynomial form function so that P(Ψ) is zero in the plasma surface and the slope of P(Ψ) at the peak is zero.

In practice, the magnetic surface test functions are unknown. Therefore it is desirable for magnetic surface functions to have generality for various function and to fit most statistically in case of using function of same form as

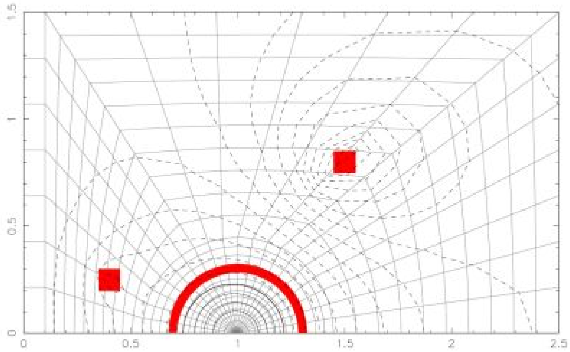

Fig.4 Example of equilibrium analysis

magnetic surface function in various function form when data fitting. So we execute $\chi 2$ examination and AIC (Akaike's Information criterion)[7] to confirm generality of magnetic surface function $P(\Psi)$ assuming the model function as follow.

$$P(\tilde{\psi}) = a\tilde{\psi} + b\tilde{\psi}^2 + \cdots + l\tilde{\psi}^p \qquad (6)$$

We assume the polynomial form to the model function for the conditions that the magnetic axis is single and this is a strictly increasing function from ($P(\tilde{\psi})$, $\tilde{\psi}$) = (0,0) to (1,1).

Each figure in Fig.5 shows a change of SSQ with one parameter of a, b, c and d in case of σ^2(=1/w_j in eq(4))=10^{-4}, 10^{-2} and 0.04, where other parameters are set to these primarily values (a=0.15, b=0, c=-0.05, d=0).

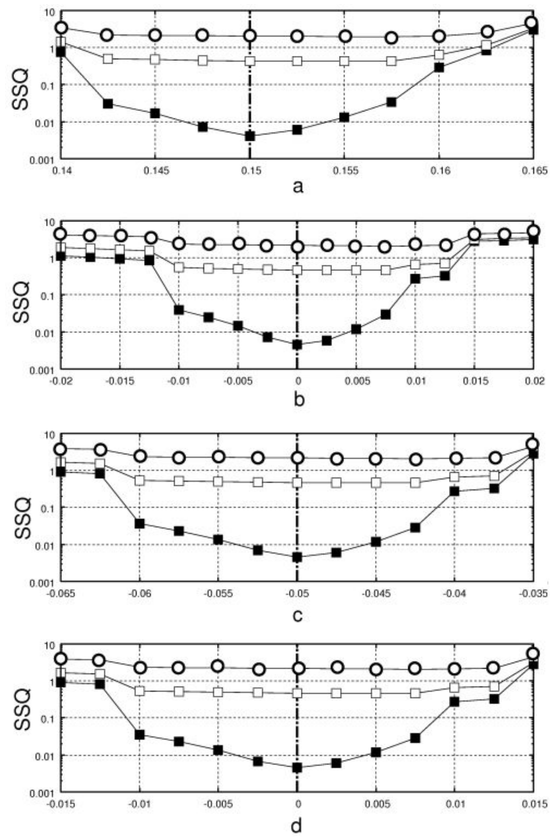

Fig.5 SSQ of Each Parameter

(■: σ^2=10^{-4}, □: σ^2=10^{-2}, ○: σ^2=0.04)

Although, the change of SSQ becomes small with increasing σ^2, SSQ has the minimum point near these primarily values up to σ^2=0.04 from the numerical comparisons.

3.1 χ^2 test

Figure 6 shows the residual SSQ in case of p=1 to 10(σ^2=0.04). This shows that the residual SSQ decrease with increasing *p* and does not decrease any further even if $p \geqq 3$. If the measurement data of n sets is y_i and the computing data by least-square method is $y(x_i)$,χ^2 is given by following equation,

$$\chi^2 = \sum_{i=1}^{n} \frac{1}{\sigma^2} \{y_i - y(x_i)\}^2 \qquad (7)$$

where σ^2 is variance. If a set of $\{y_i\}$ depends on the polynomial of p degrees shown eq.(6), χ^2 obeys a χ^2 distribution with *n-p* degrees of freedom. If χ^2 is lower than χ^2_r, which is a critical value for a significance level γ, we can estimate that the model function has conformity and so this function is allowable. Figure 6 shows the result of χ^2 test in case of γ=0.05. We can verify that model function $P(\Psi)$ is allowable in case of *p*=2,3・・,10.

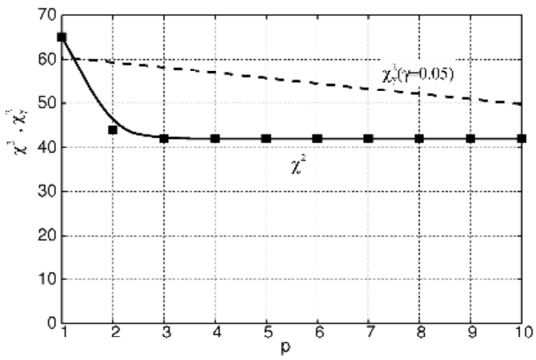

Fig.6 Result of χ^2 examination

3.2 AIC examination

The value of AIC[7] is given by

$$AIC \equiv SSQ' + 2r - n \qquad (8)$$

where SSQ' is square sum of difference with weight, r is rank, n is a number of measurement data. In AIC, if the degree increase, first term is decrease yet second term is decrease. Therefore it is expected for AIC to have minimum at a certain degree p, and then the polynomial of this p degree reflect most sufficient the structure of model.

Figure7 shows result of AIC examination. We verified that polynomial eq.(6) in case of p=3 fits most statistical for magnetic surface function $P(\Psi)$ eq.(5). In this time, we verified to fit most statistical in case of using the function of same form for magnetic surface function derived. From now on, we verified that the estimation of AIC is effective as the criterion judging which the polynomial of degree in the plural polynomial reflected most sufficient structure of object.

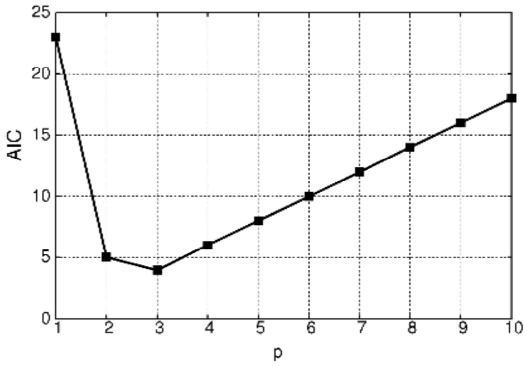

Fig.7 Result of AIC examination

4. Parallel calculation of Equilibrium Analysis

To obtain a converged solution of the N-LSM, a large number of equilibrium calculations must be performed. So, to improve the computational speed, besides a technique using the Green's functions[2], we examine the application of the parallel FEM calculation GeoFEM routine using the field partition method. The number of field partition is 1,2 and 4 and we partition the analysis region along the circumference. We assigned one CPU to one partitioned field. The solving method for solving the simultaneous linear equation in parallel linear solver is CG(Conjugate Gradient) method, preconditioning is ILU and decision of convergence is 1.0×10^{-12}. Table 1 shows the result in this case. Though iterative time is increase by increasing the number of field partition, the degree of this change do not increase so much.

In CG method, a computing time increase in proportion to 1.3 ～ 1.5th power of the number of nodes. So we need to consider the increase of computing time for the increase of iterative time and communication cost and the decrease of computing time for the decrease of the number of nodes. So, it is possible to speed up by parallel computing of field partition because iterative time doesn't change so much though the field partition increase as shown in Tab.1. We examined the calculation cost of CG method in case of not partitioning the analysis field and partitioning the analysis field into four in iterative method. In this result, we can expect speeding up about 2.8 times using 4CPU and find that this calculation is effective in the reconstruction code. From this result, the speed up ratio is estimated to be 2.8 for 4CPU in spite of relatively small number of nodes. So further speed up by using this field partition and parallelization for parameters in N-LSM can be expected[8].

Table.1 Iteration number of GeoFEM solver in each partition (PE is processing element)

	Iteration number	The number of nodes in one PE
Field partition 1	70	505
Field partition 2	75	274
Field partition 4	81	168

5. Conclusion

We have developed the code to calculate the magnetic surface function by the least square method from the measurement data for an axi-symmetric magnetic confined plasma and get the following result.

Using χ^2 and AIC examination, it is shown that the model function P(Ψ) of polynomial form corresponds with the test function of polynomial form up to σ^2=0.04

We change the solver using the massive simultaneous linear equation in finite element method code into solver of GeoFEM which supported the communication between fields and estimate the speeding up effect of calculation. As a result iterative time of iterative method do not increase so much and we confirm that it is possible to parallelize by solver of GeoFEM.

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