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A Novel Method to Construct Stationary Solutions of the Vlasov-Maxwell System

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We present a method to derive stationary solutions of the Vlasov-Maxwell system. In this method, a polynomial series is used to expand the deviation of the distribution function from the equilibrium distribution function. We use the Hermite polynomial series for classical Maxwell-Boltzmann distribution. On the other hand, we define an appropriate polynomial series for the Maxwell-Jüttner distribution, the relativistic extension of the Maxwell-Boltzmann distribution. By applying our method, one can construct various equilibrium configurations of collisionless plasmas. In particular, we find a new two-dimensional equilibrium, which may provide an initial setup for investigation of three dimensional reconnection of magnetic fields in the collisionless plasma.

Keywords: collisionless plasma, Vlasov-Maxwell system, equilibrium

1. Introduction

Plasmas in which the effect of collisions between particles composing the plasma is negligible are called collisionless plasmas. Such plasmas are considered to play key roles both in astrophysical phenomena and laboratory experiments. In particular, equilibrium configurations of collisionless plasmas are of great interest because they provide convenient initial setups for their stability analysis and investigations of collisionless magnetic reconnection. The Vlasov-Maxwell system is one of the most reliable model for collisionless plasmas. Therefore, some authors have investigated stationary solutions of the Vlasov-Maxwell system, for example, the Harris sheet [1], the Bennet pinch [2], and the BGK solution [3]. Recently, we have proposed a new method to construct stationary solutions of the Vlasov-Maxwell system. The details of the method are found in [4, 5]. In this paper, we review the method and derive the two-dimensional equilibrium configurations proposed in [4, 5].

2. Formulation

In this section, we introduce the Vlasov-Maxwell system and define two orthogonal polynomial series which are used in the next section. The Vlasov equation governs the kinetic evolution of the distribution function of particle j defined in phase space (x, y, z, p_x, p_y, p_z). Whereas, the Maxwell equations govern the evolution of the electromagnetic fields.

We consider a stationary plasma uniformly extending in the z-direction. Under this assumption, the complete set of equations of the Vlasov-Maxwell system is as follows;

$$\begin{aligned}
&\frac{p_x}{m_jc\gamma_j}\frac{\partial f_j}{\partial x}+\frac{p_y}{m_jc\gamma_j}\frac{\partial f_j}{\partial y}\\
&\quad+\frac{q_j}{c}\left(E_x-\frac{p_z}{m_jc\gamma_j}B_y\right)\frac{\partial f_j}{\partial p_x}\\
&\quad+\frac{q_j}{c}\left(E_y-\frac{p_z}{m_jc\gamma_j}B_x\right)\frac{\partial f_j}{\partial p_y}\\
&\quad+\frac{q_j}{c}\left(\frac{p_x}{m_jc\gamma_j}B_y-\frac{p_y}{m_jc\gamma_j}B_x\right)\frac{\partial f_j}{\partial p_z}=0
\end{aligned}\tag{1}$$

$$\begin{aligned}
E_x&=-\frac{\partial\phi}{\partial x},\quad E_y=-\frac{\partial\phi}{\partial y},\\
B_x&=\frac{\partial A_z}{\partial y},\quad B_y=-\frac{\partial A_z}{\partial x}
\end{aligned}\tag{2}$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=-4\pi\rho,\quad\frac{\partial^2A_z}{\partial x^2}+\frac{\partial^2A_z}{\partial y^2}=-4\pi j_z\tag{3}$$

$$\begin{aligned}
\rho&=\sum_j q_j\int_{-\infty}^{\infty}dp_x\int_{-\infty}^{\infty}dp_y\int_{-\infty}^{\infty}dp_z f_j,\\
j_z&=\sum_j q_j\int_{-\infty}^{\infty}dp_x\int_{-\infty}^{\infty}dp_y\int_{-\infty}^{\infty}dp_z\frac{p_z}{m_jc\gamma_j}f_j,
\end{aligned}\tag{4}$$

where f_j represents the momentum distribution function for particles j with the charge q_j and the mass m_j. γ_j is the Lorentz factor, which is expressed by introducing dimensionless momentum $\hat{p}_x = p_x/(m_jc), \hat{p}_y = p_y/(m_jc)$, and $\hat{p}_z = p_z/(m_jc)$ as $\gamma_j = \sqrt{1+\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2}$. c is the speed of light. E_x, E_y and B_x, B_y represent the x, y-components of the electric and magnetic fields. ϕ and A_z are the scalar potential and the z-component of the vector potential. ρ and j_z are the charge density and the z-component of the electric current density.

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Before we derive the stationary solution of the equations above, we define orthogonal polynomial series $S_n^{\rm even}$ and $S_n^{\rm odd}$, which satisfy the following conditions; (i) $S_n^{\rm even}$ are even functions of $\hat{p}_z$ and $S_n^{\rm odd}$ are odd functions of $\hat{p}_z$. (ii) The orthogonality relations;

$$\int_{-\infty}^{\infty} d\hat{p}_x \int_{-\infty}^{\infty} d\hat{p}_y \int_{-\infty}^{\infty} d\hat{p}_z S_m^{\rm even}(\hat{p}_z) S_m^{\rm even}(\hat{p}_z) \times \exp\left[-\zeta\sqrt{1+\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2}\right] \propto \delta_{mn}, \tag{5}$$

and

$$\int_{-\infty}^{\infty} d\hat{p}_x \int_{-\infty}^{\infty} d\hat{p}_y \int_{-\infty}^{\infty} d\hat{p}_z S_m^{\rm odd}(\hat{p}_z) S_m^{\rm odd}(\hat{p}_z) \times \frac{\exp\left[-\zeta\sqrt{1+\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2}\right]}{\sqrt{1+\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2}} \propto \delta_{mn}, \tag{6}$$

where ζ is an independent variable. Some expressions of them are

$$\begin{aligned}
S_0^{\rm even} =& 1,\\
S_1^{\rm even} =& \hat{p}_z^2 - \frac{K_3(\zeta)}{\zeta K_2(\zeta)},\\
S_2^{\rm even} =& \hat{p}_z^4 - \frac{3}{\zeta}\frac{5K_5(\zeta)K_2(\zeta) - K_4(\zeta)K_3(\zeta)}{3K_4(\zeta)K_2(\zeta) - [K_3(\zeta)]^2}\\
&\times\left[\hat{p}_z^2 - \frac{K_3(\zeta)}{\zeta K_2(\zeta)}\right] - \frac{2K_4(\zeta)}{\zeta^2 K_2(\zeta)},\\
S_0^{\rm odd} =& \hat{p}_z,\\
S_1^{\rm odd} =& \hat{p}_z^3 - \frac{3K_3(\zeta)}{\zeta K_2(\zeta)}\hat{p}_z,\\
S_2^{\rm odd} =& \hat{p}_z^5 - \frac{5}{\zeta}\frac{7K_5(\zeta)K_2(\zeta) - 3K_4(\zeta)K_3(\zeta)}{5K_4(\zeta)K_2(\zeta) - 3[K_3(\zeta)]^2}\\
&\times\left[\hat{p}_z^3 - \frac{3K_3(\zeta)}{\zeta K_2(\zeta)}\hat{p}_z\right] - \frac{15K_4(\zeta)}{\zeta^2 K_2(\zeta)},
\end{aligned} \tag{7}$$

where K_n represents the n-order modified Bessel function of the second kind. From the condition (i), the following integrals vanish;

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\hat{p}_x \int_{-\infty}^{\infty} d\hat{p}_y \int_{-\infty}^{\infty} d\hat{p}_z S_m^{\rm even}(\hat{p}_z) S_m^{\rm odd}(\hat{p}_z) \times \exp\left[-\zeta\sqrt{1+\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2}\right] = 0,\\
&\int_{-\infty}^{\infty} d\hat{p}_x \int_{-\infty}^{\infty} d\hat{p}_y \int_{-\infty}^{\infty} d\hat{p}_z S_m^{\rm even}(\hat{p}_z) S_m^{\rm odd}(\hat{p}_z) \times \frac{\exp\left[-\zeta\sqrt{1+\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2}\right]}{\sqrt{1+\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2}} = 0.
\end{aligned} \tag{8}$$

3. Derivation

We assume that the distribution function takes the following form,

$$f_j = \left[g_{j,0}^{\rm even}(A_z)S_0^{\rm even}(\hat{p}_z) + g_{j,0}^{\rm odd}(A_z)S_0^{\rm odd}(\hat{p}_z) + g_{j,1}^{\rm even}(A_z)S_1^{\rm even}(\hat{p}_z)\right]\exp\left(-\frac{q_j\phi}{k_{\rm B}T_j}\right) f_j^{\rm MJ}, \tag{9}$$

where $k_{\rm B}$ is the Boltzmann constant. T_j is the temperature independent of the position. $\zeta_j = m_jc^2/(k_{\rm B}T_j)$ is a dimensionless variable. $f_j^{\rm MJ}$ is the Maxwell-Jüttner distribution defined as

$$f_j^{\rm MJ} = \frac{n_j}{4\pi m_j^2 ck_{\rm B}T_jK_2(\zeta_j)} \times \exp\left[-\zeta_j\sqrt{1+\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2}\right], \tag{10}$$

where n_j is the number density of particles j. It is assumed to be constant. Although we have truncated the polynomial series for simplicity and only three polynomials $S_0^{\rm even}$,$S_0^{\rm odd}$, and $S_1^{\rm even}$ are used to expand the deviation of the distribution function from the Maxwell-Jüttner function, one can construct more complex stationary solutions by taking higher order terms.

Here we consider plasmas in charge neutrality composed of electrons and ions with the same charge but the opposite sign ($q_i = -q_e = e$ and $n_i = n_e = n_0$) in which the electric field strength is sufficiently small ($\phi = 0$). In this case one can verify that ions and electrons have the same spatial distribution

$$g_{i,0}^{\rm even} = g_{e,0}^{\rm odd} \tag{11}$$

Then, substitution of the expression (9) into the Vlasov equation leads to

$$\begin{aligned}
\frac{dg_{j,0}^{\rm even}}{dA_z} =& \frac{q_j}{m_jc^2}g_{j,0}^{\rm odd},\\
\frac{dg_{j,0}^{\rm odd}}{dA_z} =& \frac{2q_j}{m_jc^2}g_{j,1}^{\rm even},\\
\frac{dg_{j,1}^{\rm even}}{dA_z} =& 0,
\end{aligned} \tag{12}$$

A set of solutions of these equations is reduced to

$$\begin{aligned}
&g_{i,0}^{\rm even} = g_{e,0}^{\rm even} = \left(\frac{eA_z}{m_jc^2}\right)^2 + C,\\
&g_{i,0}^{\rm odd} = \frac{2eA_z}{m_ic^2}, \quad g_{e,0}^{\rm odd} = -\frac{2em_eA_z}{m_i^2c^2},\\
&g_{i,1}^{\rm even} = 1, \quad g_{e,1}^{\rm even} = \frac{m_e^2}{m_i^2},
\end{aligned} \tag{13}$$

where C is a constant. These solutions lead to the following distribution function

$$f_i = \frac{n_0}{2\pi c k_B T_i K_2(\zeta_i)} \times \left[\frac{1}{m_i^2 c^2}\left(p_z + \frac{eA_z}{c}\right)^2 + C - \frac{K_3(\zeta_i)}{\zeta_i K_2(\zeta_i)}\right] \times \exp\left[-\frac{c}{k_B T_i}\sqrt{m_i^2 c^2 + p_x^2 + p_y^2 + p_z^2}\right], \tag{14}$$

$$f_e = \frac{n_0}{2\pi c k_B T_e K_2(\zeta_e)} \times \left[\frac{1}{m_i^2 c^2}\left(p_z + \frac{eA_z}{c}\right)^2 + C - \frac{m_e^2 K_3(\zeta_i)}{m_i^2 \zeta_e K_2(\zeta_i)}\right] \times \exp\left[-\frac{c}{k_B T_e}\sqrt{m_e^2 c^2 + p_x^2 + p_y^2 + p_z^2}\right]. \tag{15}$$

Here we should note that the constant C must satisfy the following conditions in order for the distribution functions to take positive values at any point in phase space;

$$C \geq \frac{K_3(\zeta_i)}{\zeta_i K_2(\zeta_i)}, \quad C \geq \frac{m_e^2 K_3(\zeta_e)}{m_i^2 \zeta_e K_2(\zeta_e)}. \tag{16}$$

Next, we derive the configuration of magnetic fields. From equations (3) and (4), it is found that the z-component of the vector potential is determined from

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = -\frac{8\pi e^2}{c^2}\left(\frac{n_i m_i}{\zeta_i} + \frac{n_e m_e}{\zeta_e}\right) A_z, \tag{17}$$

which have a solution in the form of

$$A_z = -B_0 L[\cos(x/L) + \cos(y/L)]. \tag{18}$$

L is a constant satisfying the following relation:

$$L = \frac{c}{2e}\sqrt{\frac{\zeta_i \zeta_e}{2\pi(n_i m_i \zeta_e + n_e m_e \zeta_i)}}. \tag{19}$$

The corresponding magnetic fields are

$$B_x = B_0 \sin(y/L), \quad B_y = -B_0 \sin(x/L). \tag{20}$$

One can easily check that equations (14), (15), and (18) satisfy the relativistic Vlasov-Maxwell system exactly.

Figure 1 shows the equilibrium configuration expressed by equations (14), (15), and (18). The gray scale represents the density distribution of ions (or electrons) in arbitrary units and the arrows represent the magnetic fields. The current filaments lie along the z axis and generate the magnetic fields around themselves. We focus on the momentum distributions at the center of a filament (referred to as the O-point) and a middle point between the filaments (referred to as the X-point). Figure 2 shows the momentum distribution of ions at the X- and O-point. The values of parameters are as follows; $B_0 = 0.1 m_e \omega_e / c$, where ω_e is the electron plasma frequency, $k_B T_i = k_B T_e = m_i c^2$, and $C = 0.0005$. At the X-point, a symmetric two-peak distribution is achieved. On the other hand, an asymmetric one is achieved at the O-point.

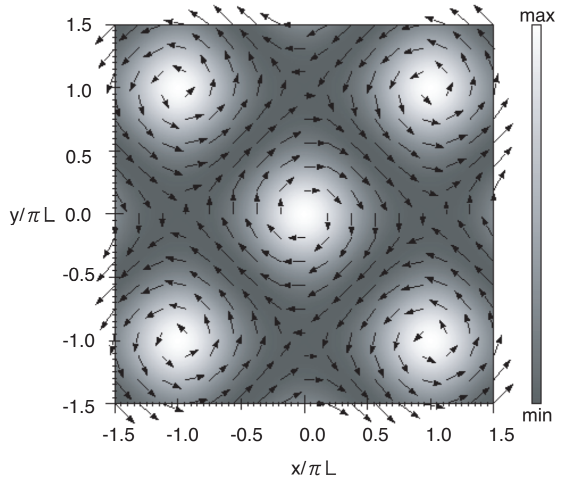

Fig. 1 The configuration of the equilibrium expressed by equations (14), (15), and (18). The gray scale represents the density distribution of ions (or electrons) in arbitrary units and the arrows represent the magnetic fields.

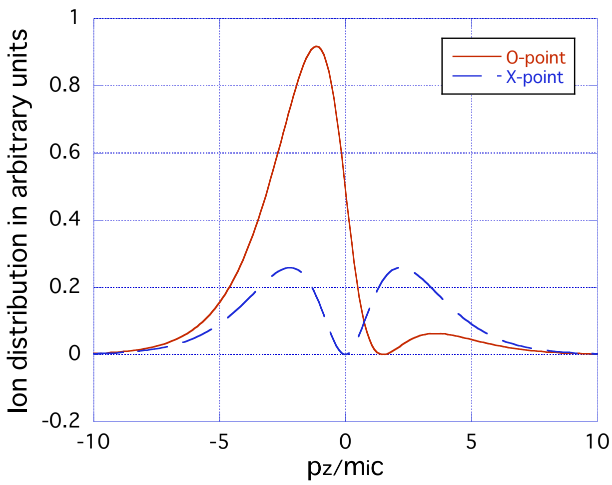

Fig. 2 the momentum distribution of ions at the O-point (solid line) and the X-point (dashed line).

4. Non-relativistic limit

In this section, we consider the non-relativistic limit of the treatment described in the previous section. In this limit ($\zeta \to \infty$) , the polynomial series $S_n^{\rm even}$ and $S_n^{\rm odd}$ can be reduced to the Hermite polynomial series as

$$S_n^{\rm even} = \frac{H_{2n}(\sqrt{2\zeta}\hat{p}_z)}{\zeta^n}, \quad S_n^{\rm odd} = \frac{H_{2n+1}(\sqrt{2\zeta}\hat{p}_z)}{\zeta^{n+1/2}}. \tag{21}$$

Therefore, we can construct stationary solutions of the non-relativistic Vlasov-Maxwell system by using a similar procedure described in the previous section. In this way, [4] succeeded in deriving the equilibria already known in the literature [1, 2] and a new two dimensional equilibrium of non-relativistic collisionless plasmas. One can verify that the following distribution function and vector potential satisfy the non-relativistic Vlasov-Maxwell system exactly,

$$\begin{aligned} f_j =& \frac{n_0}{4\pi^{5/2}L^2(m_i v_i^2 + m_e v_e^2)v_j^3} \\ &\times \left[\left(A_z + \frac{m_j c}{q_j}v_z\right)^2 + A_0^2\right] \exp\left(-\frac{v_x^2+v_y^2+v_z^2}{v_j^2}\right), \end{aligned} \tag{22}$$

and

$$A_z = -B_0 L[\cos(x/L) + \cos(y/L)]. \tag{23}$$

v_x,v_y, and v_z are the velocity coordinates corresponding to the momentum coordinates (p_x, p_y, p_z). L and A_0 are constants. v_j is the thermal velocity of particles j. This equilibrium is non-relativistic counterpart of the equilibrium expressed by (14), (15), and (18).

5. Conclusion

In this paper, we review the method to construct stationary solutions of the Vlasov-Maxwell system proposed by [4, 5] and derive a two-dimensional equilibrium configuration. It may provide a convenient initial setup for investigation of collisionless magnetic reconnection. The detailed investigation of the equilibrium, e.g., the stability analysis, should be performed in future work.

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