A Novel Method to Construct Stationary Solutions of the Vlasov-Maxwell System

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We present a method to derive stationary solutions of the Vlasov-Maxwell system. In this method, a polynomial series is used to expand the deviation of the distribution function from the equilibrium distribution function. We use the Hermite polynomial series for classical Maxwell-Boltzmann distribution. On the other hand, we define an appropriate polynomial series for the Maxwell-Jüttner distribution, the relativistic extension of the Maxwell-Boltzmann distribution. By applying our method, one can construct various equilibrium configurations of collisionless plasmas. In particular, we find a new two-dimensional equilibrium, which may provide an initial setup for investigation of three-dimensional reconnection of magnetic fields in the collisionless plasma.

Keywords: collisionless plasma, Vlasov-Maxwell system, equilibrium

1. Introduction

Plasmas in which the effect of collisions between particles composing the plasma is negligible are called collisionless plasmas. Such plasmas are considered to play key roles both in astrophysical phenomena and laboratory experiments. In particular, equilibrium configurations of collisionless plasmas are of great interest because they provide convenient initial setups for their stability analysis and investigations of collisionless magnetic reconnection. The Vlasov-Maxwell system is one of the most reliable models for collisionless plasmas. Therefore, some authors have investigated stationary solutions of the Vlasov-Maxwell system, for example, the Harris sheet [1], the Bennet pinch [2], and the BGK solution [3]. Recently, we have proposed a new method to construct stationary solutions of the Vlasov-Maxwell system. The details of the method are found in [4, 5]. In this paper, we review the method and derive the two-dimensional equilibrium configurations proposed in [4, 5].

2. Formulation

In this section, we introduce the Vlasov-Maxwell system and define two orthogonal polynomial series which are used in the next section. The Vlasov equation governs the kinetic evolution of the distribution function of particle \( j \) defined in phase space \((x, y, z, p_x, p_y, p_z)\). Whereas, the Maxwell equations govern the evolution of the electromagnetic fields.

We consider a stationary plasma uniformly extending in the \( z \)-direction. Under this assumption, the complete set of equations of the Vlasov-Maxwell system is as follows;

\[
\begin{align*}
\frac{p_x}{m_j c \gamma_j} \frac{\partial f_j}{\partial x} + \frac{p_y}{m_j c \gamma_j} \frac{\partial f_j}{\partial y} + \frac{p_z}{m_j c \gamma_j} \frac{\partial f_j}{\partial z} \\
= \frac{q_j}{c} \left( E_x - \frac{p_x}{m_j c \gamma_j} B_y \right) \frac{\partial f_j}{\partial p_x} + \frac{q_j}{c} \left( E_y - \frac{p_y}{m_j c \gamma_j} B_z \right) \frac{\partial f_j}{\partial p_y} + \frac{q_j}{c} \left( B_x \frac{p_y}{m_j c \gamma_j} - B_y \frac{p_y}{m_j c \gamma_j} \right) \frac{\partial f_j}{\partial p_z} = 0
\end{align*}
\]

\( E_x = -\frac{\partial \phi}{\partial x}, \quad E_y = -\frac{\partial \phi}{\partial y}, \quad B_x = \frac{\partial A_z}{\partial y}, \quad B_y = -\frac{\partial A_z}{\partial x} \)

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -4\pi \rho, \quad \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = -4\pi j_z
\]

\[
\rho = \sum_j \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_z f_j,
\]

\[
j_z = \sum_j \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_z \frac{p_z}{m_j c \gamma_j} f_j,
\]

where \( f_j \) represents the momentum distribution function for particles \( j \) with the charge \( q_j \) and the mass \( m_j \). \( \gamma_j \) is the Lorentz factor, which is expressed by introducing dimensionless momentum \( \tilde{p}_x = p_x/(m_j c), \tilde{p}_y = p_y/(m_j c) \), and \( \tilde{p}_z = p_z/(m_j c) \) as \( \gamma_j = \sqrt{1 + \tilde{p}_x^2 + \tilde{p}_y^2 + \tilde{p}_z^2} \). \( c \) is the speed of light. \( E_x, E_y \) and \( B_x, B_y \) represent the \( x, y \)-components of the electric and magnetic fields. \( \phi \) and \( A_z \) are the scalar potential and the \( z \)-component of the vector potential. \( \rho \) and \( j_z \) are the charge density and the \( z \)-component of the electric current density.
Before we derive the stationary solution of the equations above, we define orthogonal polynomial series $S_n^{\text{even}}$ and $S_n^{\text{odd}}$, which satisfy the following conditions: (i) $S_n^{\text{even}}$ are even functions of $\hat{p}_x$ and $S_n^{\text{odd}}$ are odd functions of $\hat{p}_x$. (ii) The orthogonality relations;

$$\int_{-\infty}^{\infty} d\hat{p}_x \int_{-\infty}^{\infty} d\hat{p}_y \int_{-\infty}^{\infty} d\hat{p}_z S_n^{\text{even}}(\hat{p}_x) S_n^{\text{even}}(\hat{p}_y) \times \exp \left[ -\zeta \sqrt{1 + \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2} \right] = 0,$$

where $\zeta$ is an independent variable. Some expressions of them are

\begin{align*}
S_0^{\text{even}} &= 1, \\
S_1^{\text{even}} &= \hat{p}_2 - \frac{K_3(\zeta)}{\zeta K_2(\zeta)}, \\
S_2^{\text{even}} &= \hat{p}_2^3 - 3 \frac{5 K_3(\zeta) K_2(\zeta)}{\zeta} - K_4(\zeta) K_3(\zeta) \\
&\quad \times \left[ \frac{K_4(\zeta)}{\zeta} - 2 K_3(\zeta) \right] \frac{\zeta}{\zeta^2 K_2(\zeta)}, \\
S_0^{\text{odd}} &= \hat{p}_2, \\
S_1^{\text{odd}} &= \hat{p}_2^3 - 3 \frac{K_3(\zeta)}{\zeta K_2(\zeta)} K_2(\zeta), \\
S_2^{\text{odd}} &= \hat{p}_2^5 - 5 \frac{7 K_3(\zeta) K_2(\zeta)}{\zeta} K_4(\zeta) K_3(\zeta) \\
&\quad \times \left[ \frac{K_4(\zeta)}{\zeta} - 3 K_3(\zeta) \right] \frac{\zeta}{\zeta^2 K_2(\zeta)}.
\end{align*}

where $K_n$ represents the $n$-order modified Bessel function of the second kind. From the condition (i), the following integrals vanish;

\begin{align*}
\int_{-\infty}^{\infty} d\hat{p}_x \int_{-\infty}^{\infty} d\hat{p}_y \int_{-\infty}^{\infty} d\hat{p}_z S_n^{\text{even}}(\hat{p}_x) S_n^{\text{odd}}(\hat{p}_y) \times \exp \left[ -\zeta \sqrt{1 + \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2} \right] &= 0, \\
\int_{-\infty}^{\infty} d\hat{p}_x \int_{-\infty}^{\infty} d\hat{p}_y \int_{-\infty}^{\infty} d\hat{p}_z S_n^{\text{even}}(\hat{p}_x) S_n^{\text{odd}}(\hat{p}_z) \times \exp \left[ -\zeta \sqrt{1 + \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2} \right] &= 0.
\end{align*}

3. Derivation

We assume that the distribution function takes the following form,

$$f_j = [g_j^{\text{even}}(A_z) S_0^{\text{even}}(\hat{p}_x) + g_j^{\text{odd}}(A_z) S_0^{\text{odd}}(\hat{p}_x)] + g_j^{\text{odd}}(A_z) S_1^{\text{even}}(\hat{p}_x)] \exp \left[ -\frac{q_j \phi}{k_B T_j} \right] f_j^{\text{MJ}}$$

where $k_B$ is the Boltzmann constant. $T_j$ is the temperature independent of the position. $\zeta = m_j c^2 / (k_B T_j)$ is a dimensionless variable. $f_j^{\text{MJ}}$ is the Maxwell-Jüttner distribution defined as

$$f_j^{\text{MJ}} = \frac{n_j}{4\pi m_j^2 c^2 k_B T_j K_2(\zeta)} \times \exp \left[ -\zeta \sqrt{1 + \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2} \right],$$

where $n_j$ is the number density of particles $j$. It is assumed to be constant. Although we have truncated the polynomial series for simplicity and only three polynomials $S_0^{\text{even}}$, $S_1^{\text{even}}$, and $S_1^{\text{odd}}$ are used to expand the deviation of the distribution function from the Maxwell-Jüttner function, one can construct more complex stationary solutions by taking higher order terms.

Here we consider plasmas in charge neutrality composed of electrons and ions with the same charge but the opposite sign $(q_i = -q_e = e$ and $n_i = n_e = n_0$) in which the electric field strength is sufficiently small ($\phi = 0$). In this case one can verify that ions and electrons have the same spatial distribution

\begin{align*}
g_j^{\text{even}} &= g_j^{\text{odd}}. \tag{11}
\end{align*}

Then, substitution of the expression (9) into the Vlasov equation leads to

\begin{align*}
\frac{d g_j^{\text{even}}}{dA_z} &= \frac{q_j}{m_j c^2} g_j^{\text{odd}}, \\
\frac{d g_j^{\text{odd}}}{dA_z} &= \frac{2 q_j}{m_j c^2} g_j^{\text{even}}, \\
\frac{d g_j^{\text{even}}}{dA_z} &= 0.
\end{align*}

A set of solutions of these equations is reduced to

\begin{align*}
g_{i,0}^{\text{even}} &= g_{e,0}^{\text{even}} = \left( \frac{e A_z}{m_j c^2} \right)^2 + C, \\
g_{i,0}^{\text{odd}} &= \frac{2 e A_z}{m_j c^2}, \quad g_{e,0}^{\text{odd}} = -\frac{2 e n_e A_z}{m_j c^2}, \tag{13}
g_{i,1}^{\text{even}} &= 1, \quad g_{e,1}^{\text{even}} = \frac{m_j^2}{m_i^2},
\end{align*}

where $C$ is a constant. These solutions lead to the
The magnetic fields. The current filaments lie along electrons) in arbitrary units and the arrows represent scale represents the density distribution of ions (or parameters are as follows; B_0 = 0.1 m_e \omega_e / c, where \omega_e is the electron plasma frequency, k_T T_1 = k_T T_e = m_e c^2, and C = 0.0005. At the X-point, a symmetric two-peak distribution is achieved. On the other hand, an asymmetric one is achieved at the O-point.

**Fig. 1** The configuration of the equilibrium expressed by equations (14), (15), and (18). The gray scale represents the density distribution of ions (or electrons) in arbitrary units and the arrows represent the magnetic fields.

**Fig. 2** the momentum distribution of ions at the O-point (solid line) and the X-point (dashed line).
4. Non-relativistic limit

In this section, we consider the non-relativistic limit of the treatment described in the previous section. In this limit ($\zeta \to \infty$), the polynomial series $S_n^{\text{even}}$ and $S_n^{\text{odd}}$ can be reduced to the Hermite polynomial series as

$$S_n^{\text{even}} = \frac{H_{2n}(\sqrt{2}\hat{p}_z)}{\zeta^n}, \quad S_n^{\text{odd}} = \frac{H_{2n+1}(\sqrt{2}\hat{p}_z)}{\zeta^{n+1/2}}. \quad (21)$$

Therefore, we can construct stationary solutions of the non-relativistic Vlasov-Maxwell system by using a similar procedure described in the previous section. In this way, [4] succeeded in deriving the equilibria already known in the literature [1, 2] and a new two dimensional equilibrium of non-relativistic collisionless plasmas. One can verify that the following distribution function and vector potential satisfy the non-relativistic Vlasov-Maxwell system exactly,

$$f_j = \frac{n_0}{4\pi^{3/2}L^2(m_i v_i^2 + m_e v_e^2)^{3/2}} \times \left( A_z + \frac{m_e c}{q_j} v_z \right)^2 + A_0^2 \exp \left( -\frac{v_x^2 + v_y^2 + v_z^2}{v_j^2} \right), \quad (22)$$

and

$$A_z = -B_0 L \left[ \cos(x/L) + \cos(y/L) \right]. \quad (23)$$

$v_x, v_y$, and $v_z$ are the velocity coordinates corresponding to the momentum coordinates ($p_x, p_y, p_z$). $L$ and $A_0$ are constants. $v_j$ is the thermal velocity of particles $j$. This equilibrium is non-relativistic counterpart of the equilibrium expressed by (14), (15), and (18).

5. Conclusion

In this paper, we review the method to construct stationary solutions of the Vlasov-Maxwell system proposed by [4, 5] and derive a two-dimensional equilibrium configuration. It may provide a convenient initial setup for investigation of collisionless magnetic reconnection. The detailed investigation of the equilibrium, e.g., the stability analysis, should be performed in future work.