Parametric Studies of High- and Low-Frequency Magnetosonic Waves and Ion Acceleration in Two-Ion-Species Plasmas

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The propagation of low- and high-frequency magnetosonic waves and associated ion acceleration in two-ion-species plasmas are studied theoretically and numerically. It is analytically shown that the KdV equation for the low-frequency mode is valid for the amplitudes $\varepsilon < 2\Delta_\omega$, where $\Delta_\omega = (\omega_{+0} - \omega_{-r})/\omega_{+0}$ with ω_{+0} the cut-off frequency of the high-frequency mode and ω_{-r} is the resonance frequency of the low-frequency mode. With electromagnetic particle simulations, the evolution of the low- and high-frequency-mode pulses with amplitudes of the order of $2\Delta_\omega$ is investigated for H-T and D-T plasmas. It is shown that high-frequency mode pulses are generated from a long-wavelength, low-frequency-mode pulse if its amplitude exceeds $2\Delta_\omega$. Furthermore, the shock waves with amplitudes much greater than $2\Delta_\omega$ are simulated. In the cases of $n_{\rm H} = 10n_{\rm T}$ and $n_{\rm D} = n_{\rm T}$, the high-frequency-mode shock wave is generated and causes ion energization. In the case of $n_{\rm H} = n_{\rm T}$, for which Δ_ω is much greater than those for the previous two cases, however, high- and low-frequency-mode shock waves coexist and contribute to the acceleration.

Keywords: magnetosonic wave, two-ion-species plasma, soliton, shock wave, particle acceleration, particle simulation

1. Introduction

Particle simulations have shown [1] that in a multi-ion-species plasma with the hydrogen (H) being the major ion component, a magnetosonic shock wave accelerates with the transverse electric field all the heavy ions to nearly the same speed

$$v_{\rm max} \sim v_{\rm sh}(B_M - B_0)/(B_M + B_0), \qquad (1)$$

where B_0 is the strength of the external magnetic field, B_M is the peak value of the magnetic field in the shock wave, and $v_{\rm sh}$ is the shock propagation speed. This result accounts for the observation that the elemental compositions of energetic heavy ions produced in solar flares are similar to that of the ions in the solar corona [2].

In fusion plasmas, which also contain multiple ion species such as deuterium (D) and tritium (T), however, the ion densities are of the same order of magnitudes, and so are the ion masses; these differ from the circumstances of space plasmas. It is thus important to study the effect of mass and density ratios on nonlinear propagation of magnetosonic waves [3] and associated ion acceleration [4].

In this paper, we study two types of perpendicular magnetosonic waves in a two-ion-species plasma, i.e., low- and high-frequency modes. We point out the importance of the normalized frequency difference $\Delta_\omega = (\omega_{+0}-\omega_{-r})/\omega_{+0}$, where ω_{+0} is the cut-off frequency of the high-frequency mode, and ω_{-r} is the resonance frequency of the low-frequency mode; Δ_ω is determined by the density ratio and gyro frequency ratio of the two ion species. In Sec. 2, we analytically show that the Korteweg-de Vries (KdV) equation for the low-frequency mode is valid for the amplitudes $\varepsilon < 2\Delta_\omega$. Applying the result of Ref. [5], we suggests that nonlinear coupling between the low- and high-frequency modes can occur for $\varepsilon > 2\Delta_\omega$. In Sec. 3, with electromagnetic particle simulations, we investigate the nonlinear evolution of a long-wavelength sinusoidal disturbance of the low-frequency mode in H-T and D-T plasmas. The amplitude of the initial disturbance is taken to be $\delta B_z/B_0 = 0.1$. In the case of $n_{\rm H} = n_{\rm T}$, for which $2\Delta_\omega$ is greater than $\delta B_z/B_0$, the initial disturbance evolves into low-frequency-mode pulses with amplitudes $\varepsilon < 2\Delta_\omega$, and high-frequency-mode pulses are not found. In the cases of $n_{\rm H} = 10n_{\rm T}$ and of $n_{\rm D} = n_{\rm T}$, however, the amplitudes of low-frequency-mode pulses exceed $2\Delta_\omega$, and high-frequency-mode pulses are generated.

In Sec. 4, ion acceleration by shock waves with amplitudes much greater than $2\Delta_\omega$ is studied with particle simulations. In the H-T plasma with $n_{\rm H} = 10n_{\rm T}$ and in the D-T plasma with $n_{\rm D} = n_{\rm T}$, Δ_ω is small, and the high-frequency-mode shock wave is generated from a strong disturbance. In the case of $n_{\rm H} = 10n_{\rm T}$, some of the H ions are reflected at the shock front. Although the T ions are not reflected, all of them gain energies from the transverse electric field. In the case of $n_{\rm D} = n_{\rm T}$, some of the D ions are reflected from the shock front. Even though the D-to-T gyrofrequency ratio, $\Omega_{\rm D}/\Omega_{\rm T} = 1.5$, is rather close to unity,

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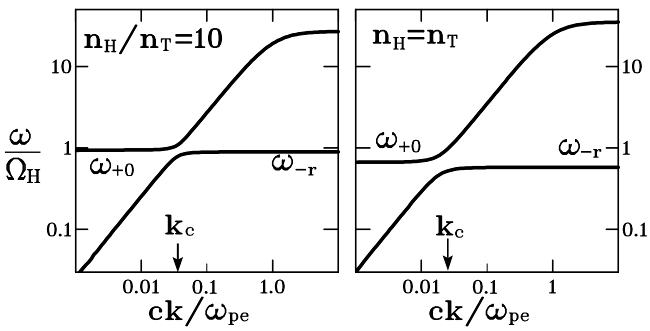

Fig. 1 Dispersion curves for high- and low-frequency modes in H-T plasmas with $n_{\rm H}/n_{\rm T} = 10$ and with $n_{\rm H} = n_{\rm T}$.

all the T ions are accelerated by the transverse electric field. For the H-T plasma with $n_{\rm H} = n_{\rm T}$, Δ_ω is much greater, and both the low- and high-frequency-mode shock waves are generated. Some of the H ions are accelerated at the shock front of the high-frequency mode. All the T ions are accelerated by the transverse electric field in both the high- and low-frequency modes; some of them further gain energies owing to the reflection by the low-frequency mode. Section 5 gives a summary of our work.

2. Two magnetosonic waves

We discuss the effect of ion composition on magnetosonic waves propagating perpendicular to a magnetic field. Figure 1 shows the dispersion curves of the low- and high-frequency modes in H-T plasmas for the cases of $n_{\rm H}/n_{\rm T} = 10$ and of $n_{\rm H}/n_{\rm T} = 1$. The two curves for the low- and high-frequency modes are close near the wave number k_c defined as

$$k_{\rm c} = \omega_{-\rm r}/v_{\rm A}, \tag{2}$$

where $v_{\rm A}$ is the Alfvén speed, $v_{\rm A} = B_0/(4\pi\rho_0)^{1/2}$ with ρ_0 the equilibrium mass density. The distance between the two curves near k_c is greater when $n_{\rm H}/n_{\rm T} = 1$ than when $n_{\rm H}/n_{\rm T} = 10$. This occurs because the frequency difference $(\omega_{+0} - \omega_{-\rm r})$ is greater in the former case than in the latter. If we denote the two ion species by a and b, then ω_{+0} and ω_{-r} [6] are given as

$$\omega_{+0} = \left(\frac{\omega_{pa}^2}{\Omega_a^2} + \frac{\omega_{pb}^2}{\Omega_b^2}\right)\frac{\Omega_a\Omega_b|\Omega_e|}{\omega_{pe}^2}, \tag{3}$$

and

$$\omega_{-\rm r} = \left(\frac{\omega_{pa}^2\Omega_b^2 + \omega_{pb}^2\Omega_a^2}{\omega_{pa}^2 + \omega_{pb}^2}\right)^{1/2}, \tag{4}$$

where, Ω_j and ω_{pj} ($j = a$ or $j = b$) represent the cyclotron and plasma frequencies, respectively.

The frequency difference normalized to ω_{+0},

$$\Delta_\omega \equiv (\omega_{+0} - \omega_{-\rm r})/\omega_{+0}, \tag{5}$$

is a function of Ω_a/Ω_b and $n_bq_b/(n_aq_a)$,

$$\Delta_\omega = 1 - (\Omega_b/\Omega_a)^{1/2}[1 + n_bq_b/(n_aq_a)] \times \left[\left(\frac{\Omega_b}{\Omega_a} + \frac{n_bq_b}{n_aq_a}\right)\left(1 + \frac{n_bq_b\Omega_b}{n_aq_a\Omega_a}\right)\right]^{-1/2}. \tag{6}$$

If $n_bq_b/(n_aq_a)$ is fixed, Δ_ω increases with increasing Ω_a/Ω_b. If Ω_a/Ω_b is fixed, Δ_ω takes its maximum value at $n_bq_b/(n_aq_a) = 1$.

In the long-wavelength region such that $k \ll k_c$, the dispersion relation of the low-frequency mode is approximated by

$$\omega = v_{\rm A}k(1 - d_l^2k^2/2), \tag{7}$$

where d_l is defined as

$$d_l^2 = \frac{v_{\rm A}^6}{c^4}\left[\frac{\omega_{pa}^2\omega_{pb}^2}{\Omega_a^2\Omega_b^2}\left(\frac{1}{\Omega_a} - \frac{1}{\Omega_b}\right)^2 + \frac{\omega_{pb}^2\omega_{\rm pe}^2}{\Omega_b^2\Omega_{\rm e}^2}\left(\frac{1}{\Omega_b} - \frac{1}{\Omega_{\rm e}}\right)^2 + \frac{\omega_{\rm pe}^2\omega_{pa}^2}{\Omega_{\rm e}^2\Omega_a^2}\left(\frac{1}{\Omega_{\rm e}} - \frac{1}{\Omega_a}\right)^2\right]. \tag{8}$$

If $n_a \sim n_b$, the length d_l is of the order of $c/(\omega_{pa}\omega_{pb})^{1/2}$. In a single-ion-species plasma, d_l reduces to $c/\omega_{\rm pe}$.

In the range of wave numbers $(m_e/m_a)^{1/2} \ll c^2k^2/\omega_{pe}^2 \ll 1$, the dispersion relation of the high-frequency mode is approximated by [7]

$$\omega = v_hk\ [1 - c^2k^2/(2\omega_{pe}^2)], \tag{9}$$

where v_h is defined as

$$v_h = v_{\rm A}\left[1 + \frac{\omega_{pa}^2\omega_{pb}^2}{\omega_{\rm pe}^4}\Omega_e^2\left(\frac{1}{\Omega_a} - \frac{1}{\Omega_b}\right)^2\right]^{1/2}. \tag{10}$$

As expected from Eq. (9), nonlinear evolution of the high-frequency mode is governed by the KdV equation [7]. In fact, the KdV equation was derived for the waves with $(m_e/m_i)^{1/2} \ll \varepsilon \ll 1$. The characteristic width of the solitary pulses of the high-frequency mode is the electron skin depth $c/\omega_{\rm pe}$. Also, Eq. (7) suggests that the nonlinear behavior of the low-frequency mode is described by the KdV equation. The characteristic width of the solitary pulses of the low-frequency mode is given by d_l, which is $\sim (m_i/m_e)$ times as long as that of the high-frequency mode.

From the facts that Eq. (7) is obtained under the assumption that $k \ll k_c$ and that the characteristic wave number of the solitary wave is $k \sim \varepsilon^{1/2}/d_l$, one finds that the KdV equation for the low-frequency mode is valid for the amplitudes $\varepsilon \ll k_c^2d_l^2$. With the aid of Eqs. (2)-(8), $k_c^2d_l^2$ is expressed as $2\Delta_\omega$. If

$$\varepsilon > 2\Delta_\omega, \tag{11}$$

however, the KdV equation is not valid. In this case, nonlinear coupling between the low- and high-frequency modes occurs [3, 5]. That is, high-frequency-mode pulses are produced from a low-frequency-mode pulse.

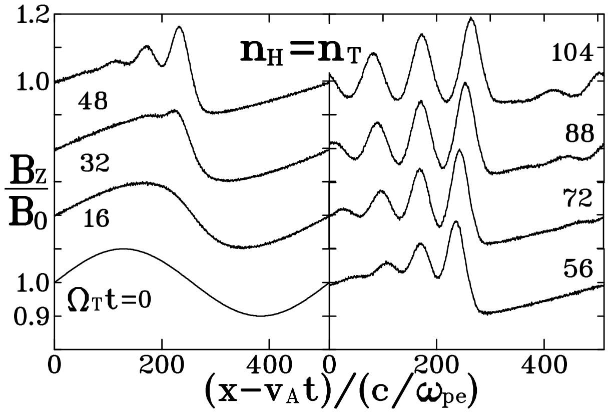

Fig. 2 Wave evolution of the low-frequency mode in H-T plasma with $n_{\mathrm{H}} = n_{\mathrm{T}}$. From the sinusoidal perturbation, low-frequency-mode pulses are formed.

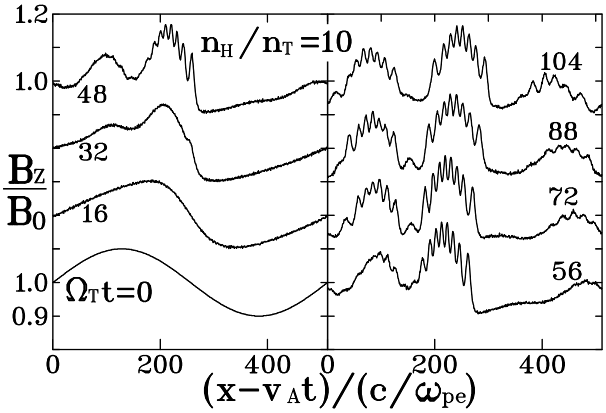

Fig. 3 Wave evolution in H-T plasma with $n_{\mathrm{H}} = 10n_{\mathrm{T}}$. From the perturbation of the low-frequency mode, the pulses of the low- and high-frequency modes are formed.

3. Nonlinear wave evolution

We study nonlinear evolution of waves using a one-dimensional (one space and three velocity components), fully kinetic, electromagnetic particle code. The system size is $L_x = 2048\Delta_{\mathrm{g}}$, where Δ_{g} is the grid spacing, with periodic boundary conditions. The total number of electrons is $N_{\mathrm{e}} \simeq 4.2\times10^6$. We consider H-T and D-T plasmas. The hydrogen-to-electron mass ratio is taken to be $m_{\mathrm{H}}/m_{\mathrm{e}} = 50$. The speed of light is $c/(\omega_{\mathrm{pe}}\Delta_{\mathrm{g}}) = 4$. The electron thermal velocity is $(T_{\mathrm{e}}/m_{\mathrm{e}})^{1/2}/(\omega_{\mathrm{pe}}\Delta_{\mathrm{g}}) = 0.5$. The ion-to-electron temperature ratio is $T_i/T_e = 0.1$; we have chosen this small ratio to avoid ion reflection by nonlinear waves. The external magnetic field is in the z direction; its strength is $|\Omega_{\mathrm{e}}|/\omega_{\mathrm{pe}} = 1.0$.

Initially, we have a sinusoidal disturbance of the low-frequency mode with a wavelength L_x propagating in the positive x direction. The amplitude of the magnetic field B_z is chosen to be $\delta B_z/B_0 = 0.1$; other components of the disturbance are determined according to the linear theory of the low-frequency mode. We observe the nonlinear evolution of this disturbance.

First, we simulate an H-T plasma with $n_{\mathrm{H}} = n_{\mathrm{T}}$. The value of Δ_ω for this case is 0.14; hence, $\delta B_z/B_0 < 2\Delta_\omega$. Figure 2 shows magnetic-field profiles at various times. As a result of nonlinear evolution, some pulses, which have shorter wavelengths than the original disturbance, are formed. These are the low-frequency-mode pulses. Even though the maximum pulse amplitude ε is greater than the amplitude of the initial disturbance, it does not exceed $2\Delta_\omega$. The wavenumbers of the low-frequency-mode pulses are $ck/\omega_{\mathrm{pe}} \sim 0.07$, which are smaller than k_c for this case, $ck_c/\omega_{\mathrm{pe}} = 0.12$.

Next, we consider an H-dominant plasma with $n_{\mathrm{H}}/n_{\mathrm{T}} = 10$. The value of Δ_ω for this case is 0.04; hence, $\delta B_z/B_0 > 2\Delta_\omega$. From the evolution of B_z in Fig. 3, we find that after the steepening of the initial sinusoidal wave of the low-frequency mode, shorter wavelength ($\sim 100c/\omega_{\mathrm{pe}}$) low-frequency-mode pulses are formed, and high-frequency-mode pulses with much shorter wavelengths ($\sim 10c/\omega_{\mathrm{pe}}$) are generated. The amplitudes of the low-frequency-mode pulses are much larger than $2\Delta_\omega$; for instance, at $\Omega_{\mathrm{H}}t = 32$, the amplitude of the right low-frequency-mode pulse is $\varepsilon = 0.27$. If $\varepsilon > 2\Delta_\omega$, high-frequency-mode pulses are generated. At $\Omega_{\mathrm{H}}t \simeq 32$, the wavenumber of the right low-frequency-mode pulse becomes almost equal to k_c, and the high-frequency mode pulses with their wavenumbers greater than k_c start to be formed.

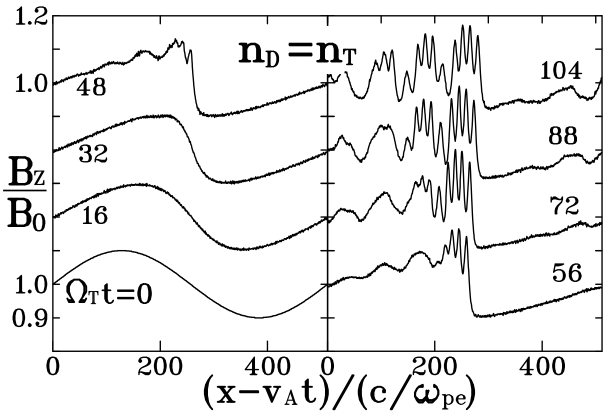

Fig. 4 Wave evolution in D-T plasma with $n_{\mathrm{D}} = n_{\mathrm{T}}$. The pulses of the low- and high-frequency modes are formed.

The normalized frequency difference Δ_ω depends also on the ion gyrofrequency ratio. Figure 4 shows the evolution of B_z in a D-T plasma with $n_{\mathrm{D}} = n_{\mathrm{T}}$. In this case, $2\Delta_\omega$ (=0.04) is smaller than the initial amplitude $\delta B_z/B_0 = 0.1$. Even though in the H-T plasma with $n_{\mathrm{H}} = n_{\mathrm{T}}$ (Fig. 2), only low-frequency-mode pulses appeared, we find in Fig. 4 many high-

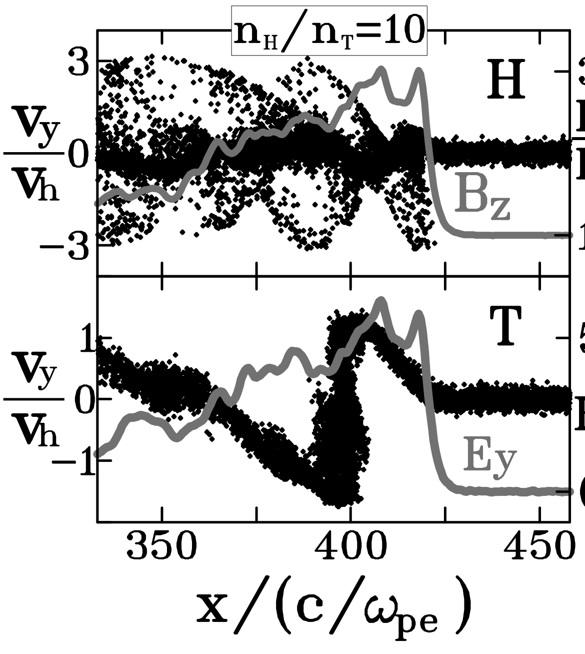

Fig. 5 Ion phase space plots (x, v_y) in H-T plasma with $n_{\rm H} = 10n_{\rm T}$ at $\omega_{\rm pe}t = 640$.

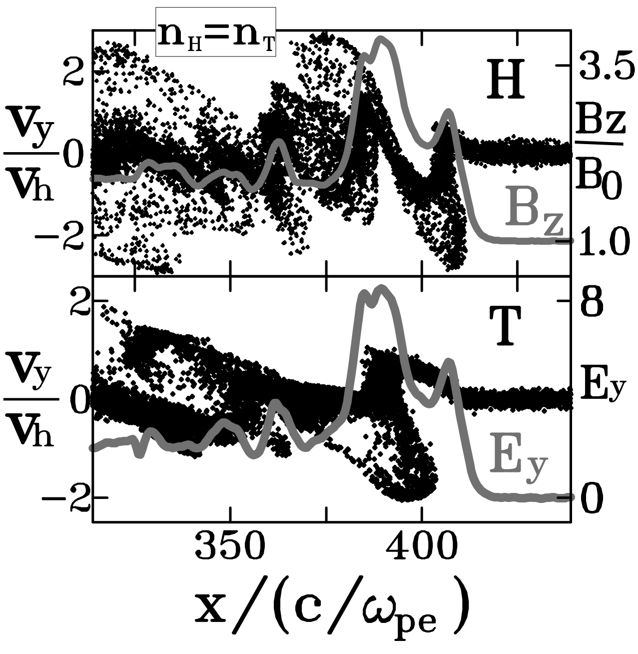

Fig. 6 Ion phase space plots (x, v_y) in H-T plasma with $n_{\rm H} = n_{\rm T}$ at $\omega_{\rm pe}t = 640$.

frequency-mode pulses with fairly large amplitudes, in addition to low-frequency-mode pulses. This difference arises from the fact that Δ_ω in D-T plasmas, where $\Omega_{\rm D}/\Omega_{\rm T} = 1.5$, is smaller than that in H-T plasmas, where $\Omega_{\rm H}/\Omega_{\rm T} = 3$.

4. Ion acceleration by shock wave

We now investigate ion acceleration by large-amplitude shock waves, using a three-dimensional, particle code. The total simulation grid size is $L_x \times L_y \times L_z = 2048\Delta_g \times 64\Delta_g \times 128\Delta_g$. The system is periodic in the y and z directions and is bounded in the x direction. The total number of simulation electrons is $N_{\rm e} \simeq 1.1 \times 10^9$. The ion-to-electron temperature ratio is $T_i/T_e = 1.0$. The external magnetic field is in the z direction with its strength being $|\Omega_{\rm e}|/\omega_{\rm pe} = 1.5$. Other parameters are the same as those in Sec. 3. The method of the shock simulation is described in detail in Ref. [1].

We present the results of three different simulation runs for H-T plasmas with $n_{\rm H}/n_{\rm T} = 10$ and 1 and for a D-T plasma with $n_{\rm D}/n_{\rm T} = 1$. In the three runs, the amplitudes of shock waves are in the range $3 < \epsilon < 4$, which are much greater than $2\Delta_\omega$ for these plasmas.

Figure 5 shows ion phase space plots (x, v_y) at $\omega_{\rm pe}t = 640$ for a shock wave with $v_{\rm sh} = 2.9v_{\rm h}$ in the H dominant plasma with $n_{\rm H}/n_{\rm T} = 10$. For this case, $\Delta_\omega = 0.04$. The solid lines represent the profiles of B_z and E_y averaged over L_y and L_z. The B_z and E_y rise sharply in the region $418 < x/(c/\omega_{\rm pe}) < 425$. Its width is of the order of $c/\omega_{\rm pe}$, which indicates that the shock wave is the high-frequency mode; although the low-frequency-mode shock wave was generated from the initial disturbance, it was quickly damped owing to the coupling with the high-freqneucy mode. The high-frequency-mode shock wave reflects some of the H ions at the shock front, $x \simeq 418(c/\omega_{\rm pe})$, and energizes them. The T ions are not reflected. All of them are, however, accelerated by the transverse electric field E_y in the positive y direction in the region $400 < x/(c/\omega_{\rm pe}) < 425$, which is much wider than the shock transition region. At $x/(c/\omega_{\rm pe}) \simeq 400$, the v_y's of the heavy ions reach their maximum values, $v_y \simeq 1.2v_{\rm h}$ ($\simeq 0.4v_{\rm sh}$). The theory (1) gives approximately the same value $v_{\max} \simeq 1.3v_{\rm h}$, which we have estimated by substituting the observed values of $v_{\rm sh}$ and B_M in Eq. (1).

For the case of $n_{\rm H} = n_{\rm T}$, the value of Δ_ω is 0.14 and is much greater than that in the previous H dominant case. Therefore, the effects of the low-frequency mode can be significant; the low-frequency mode shock wave can survive for a long time because the nonlinear coupling between high- and low-frequency modes is weak in this plasma. Figure 6 shows the field profiles and ion phase spaces for a shock wave with $v_{\rm sh} \simeq 3.2v_{\rm h}$ in a plasma with $n_{\rm H} = n_{\rm T}$. Note that both low- and high-frequency-mode shock waves exist. We find two peaks of B_z in the region $390 < x/(c/\omega_{\rm pe}) < 420$ at $\omega_{\rm pe}t = 640$. The right peak at $x/(c/\omega_{\rm pe}) \simeq 410$ is due to the high-frequency mode, while the left higher peak at $x/(c/\omega_{\rm pe}) \simeq 390$ is due

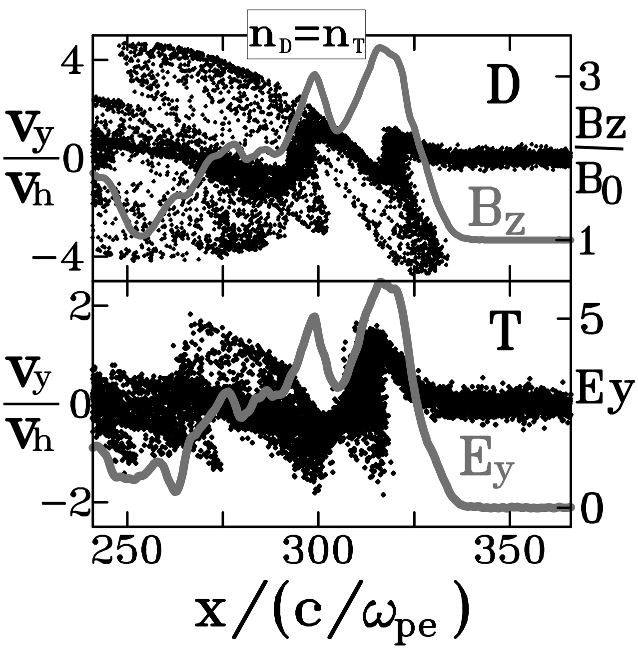

Fig. 7 Ion phase space plots (x, v_y) in D-T plasma with $n_{\rm D} = n_{\rm T}$ at $\omega_{\rm pe}t = 600$.

to the low-frequency mode. The length of the transition region of the low-frequency-mode shock wave, $25c/\omega_{\rm pe}$, is of the order of $c/(\omega_{\rm pT}\omega_{\rm pH})^{1/2}$ $(= 13c/\omega_{\rm pe})$.

As in the H dominant case, the acceleration of the H ions is caused by the reflection from the shock front of the high-frequency mode, which is at $x/(c/\omega_{\rm pe}) \simeq 400$ at $\omega_{\rm pe}t = 640$. In contrast, the T ions are energized by both the high- and low-frequency modes. The v_y's of all the T ions increase up to $v_{\rm max} \simeq 1.1v_{\rm h}$ owing to E_y in the high- and low-frequency modes. The v_x's of the T ions then increase because of the $\boldsymbol{v} \times \boldsymbol{B}$ force, which makes the relative velocities between the shock wave and the T ions small. As a result, some of them are reflected by the low-frequency-mode shock wave.

Finally, we present the result of the D-T plasma with $n_{\rm D} = n_{\rm T}$. The value of Δ_ω for this case, 0.02, is smaller than that for the $n_{\rm H}/n_{\rm T} = 10$ case, which indicates that in this D-T plasma, a high-frequency-mode shock wave is generated while the low-frequency-mode shock wave is quickly damped because of the coupling with the high-frequency mode. Figure 7 shows the wave profiles and ion phase spaces (x, v_y) at $\omega_{\rm pe}t = 640$ for the high-frequency-mode shock wave with $v_{\rm sh} \simeq 3.5v_{\rm h}$. The width of the transition region, $310 < x/(c/\omega_{\rm pe}) < 323$, is greater than that in the $n_{\rm H}/n_{\rm T} = 10$ case, even though both of them are the high-frequency mode. This broadening is caused by the ion reflection; compared with Fig. 5, there are a large number of reflected ions, and since they are the *D* ions, their gyroradii are large.

Even though the ratio of the gyrofrequencies, $\Omega_{\rm D}/\Omega_{\rm T} = 1.5$, is rather close to unity, all the T ions gain energy from E_y in the region $310 < x/(c/\omega_{\rm pe}) < 325$. Furthermore, the theory (1) gives a good estimate of the maximum speed. The observed speed, $v_y \simeq 1.4v_{\rm h} = 0.4v_{\rm sh}$, is 73% of the theoretical estimate, $v_{\rm max} \simeq 1.93v_{\rm h} = 0.55v_{\rm sh}$.

5. Summary

We have studied the effect of ion composition on the propagation of magnetosonic waves and ion acceleration in two-ion-species plasmas. First, we have analytically shown that the KdV equation for the low-frequency mode is valid for amplitudes $\varepsilon < 2\Delta_\omega$. Next, with electromagnetic particle simulations, we investigated the nonlinear evolution of a long-wavelength sinusoidal disturbance of the low-frequency mode with an initial amplitude 0.1 in H-T and D-T plasmas. In the case of $n_{\rm H} = n_{\rm T}$, the initial disturbance evolves into low-frequency-mode pulses with $\varepsilon < 2\Delta_\omega$. In the case of $n_{\rm H} = 10n_{\rm T}$ and $n_{\rm D} = n_{\rm T}$, however, the low-frequency-mode pulses with $\varepsilon > 2\Delta_\omega$ are formed, from which high-frequency-mode pulses are generated.

We have also investigated ion acceleration by large-amplitude $(\varepsilon \gg 2\Delta_\omega)$ shock waves with particle simulations. For the H-T plasma with $n_{\rm H} = 10n_{\rm T}$ and for the D-T plasma with $n_{\rm D} = n_{\rm T}$, Δ_ω is small, and the high-frequency-mode shock wave is generated from a strong disturbance and plays a central role in ion energization process. For the H-T plasma with $n_{\rm H} = n_{\rm T}$, where Δ_ω is much greater, both the low- and high-frequency-mode shock waves propagate and contribute to the ion acceleration.

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