Dark Solitons in Gravitational Wave and Pulsar Plasma Interaction

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The nonlinear propagation of gravitational wave perpendicular to a superstrong magnetic field immersed in electron-positron pulsar plasma is considered. It is found that a weak gravitational wave resonates an effective field perturbation in the strongly magnetized pulsar plasma. The slowly varying field amplitude obeys the nonlinear Schrödinger equation, the solution of which is the dark soliton whose amplitude may be quite significant in the astrophysical context.

Keywords: gravitational wave, pulsar, neutron star, black hole, electron-positron plasma, dark soliton.

1. Introduction

Study of gravitational wave interaction with strongly magnetized plasma attains much attention, largely due to increased possibility of its direct detection by facilities such as LIGO (Laser Interferometer Gravitational Wave Observatory), or by LISA (Laser Interferometer Space Antenna) [1]. The interaction of gravitational wave with plasma and excitation of electromagnetic radiation lead to alternative proposal for gravitational wave detectors [2]. Gravitational wave-plasma interaction has been studied during eighties by a group of the Kazan School of gravitation (see for example Ref. [3] and references therein) and later by Marklund *et. al*. [4]. It was shown that parametric excitation of high frequency plasma waves by gravitational radiation may take place in a magnetized plasma. In an astrophysical context, gravitational waves often propagate in a plasma medium, and the amplitude can be much larger. Källberg *et. al*. [2] considered coupled gravitational and electromagnetic waves propagating in a magnetized plasma, with the direction of propagation perpendicular to the background magnetic field. For the wave coupling to be efficient, the interaction is supposed to be almost resonant, i.e. the electromagnetic wave propagation velocity in the plasma should be close to the speed of light. This increases the interaction strength and allows neglecting the effect of the background curvature, in comparison with the direct interaction with matter.

Therefore explicit nonlinearities from the Einstein tensor can be neglected keeping the nonlinearities associated with the energy-momentum tensor (i.e. the matter and field evolution). The nonlinear gravitational wave–plasma interaction has been studied in recent time. The nonlinear response gives raise to effects such as parametric instabilities, large density fluctuations and photon acceleration [5]

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In this paper, we consider the nonlinear interaction between gravitational and electromagnetic waves in a strongly magnetized pulsar plasma. The gravitational wave is considered to propagate perpendicular to the superstrong pulsar magnetic field. Under WKB approximation to the wave equation, the slowly varying amplitude obeys the well known nonlinear Shrödinger equation (NLSE) [6]. From the analysis of group dispersion and nonlinear frequency shift it is found that the envelope is a dark soliton whose amplitude may be quite significant in a super strong magnetic field which might have some astrophysical significances.

2. Coupled Gravito-Plasma Wave Equations

The interaction between gravitational wave and plasma is governed, by the Einstein field equations

$$G_{ab} = kT_{ab} \tag{1}$$

and the Maxwell field equation

$$\nabla_a F^{ab} = j^b , \tag{2}$$

$$\nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} = 0 \quad , \tag{3}$$

where G_{ab} is the Einstein tensor, T_{ab} is the energy-momentum tensor; F_{ab} is the electromagnetic field tensor, J^b is the four current density and ∇ denotes covariant differentiation. Here, $k = 8\pi G$, $c = \mu_0 = \varepsilon_0 = 1$ and space like signature $(-+++)$ for the metric is used.

The plasma is seen as a number of charged fluids, one for each plasma species, and collisions between particles are neglected. In the absence of collisions, the evolution equations for each fluid species can be written as

$$\nabla_b T^{ab}_{(s)} = F^{ab} j_{b(s)} . \quad (4)$$

For an observer with four-velocity V^a, electromagnetic field can be decomposed into an electric part $E_a = F_{ab}V^b$ and magnetic part $B_a = 0.5\varepsilon_{abc}F^{bc}$, where $\varepsilon_{abc} = V^d \varepsilon_{abcd}$. Here ε_{abcd} is the $4-$dimensional volume element with $\varepsilon_{0123} = \sqrt{|\det g_{ab}|}$. Then an orthonormal frame (ONF) is introduced with basis $\{\vec{e}_\alpha = e^\mu_\alpha \partial_{x^\mu}\}$, where $\vec{e}_0 = \vec{V} = V^\alpha e_\alpha$, (i.e. $V^\alpha = \delta^\alpha_0$). The fluid four velocity is written as $u^\alpha = (\gamma, \gamma\vec{v})$, where $\gamma = (1 - v_\alpha v^\alpha)^{-1/2}$, $\alpha = 1,2,3$ and $\vec{v}$ is the fluid three velocity. Defining the four current $j^\alpha = \sum_s q_{(s)} n_{(s)} u^\alpha_{(s)}$, the Maxwell equations (2) and (3) and the fluid equation (4) can be written as [7]

$$\nabla \cdot \vec{E} = \rho + \rho_E , \quad (5)$$

$$\nabla \cdot \vec{B} = \rho_B , \quad (6)$$

$$e_0 \vec{E} - \nabla \times \vec{B} = -\vec{j} - \vec{J}_E , \quad (7)$$

$$e_0 \vec{B} + \nabla \times \vec{E} = -\vec{j}_B , \quad (8)$$

$$e_0(\gamma n) + \nabla \cdot (\gamma n \vec{v}) = \Delta n , \quad (9)$$

$$\begin{aligned}&(\mu + p)(e_0 + \vec{v}\cdot\nabla)\gamma\vec{v} = \\ &-\gamma^{-1}\nabla p - \gamma\vec{v}(e_0 + \vec{v}\cdot\nabla)p \\ &+ qn(\vec{E} + \vec{v}\times B) + (\mu + p)\vec{g}\end{aligned} , \quad (10)$$

where the fluid species index s is omitted. Here $\vec{E} \equiv (E^\alpha) = (E^1, E^2, E^3)$ etc. and $\nabla \equiv (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ are three vectors. The charge density is $\rho = \sum q_s \gamma n_{(s)}$ and n is proper particle number density. The effective charges, currents and forces, originating from the inclusion of gravitational field, are given by

$$\rho_E \equiv -\Gamma^\alpha_{\beta\alpha} E^\beta - \varepsilon^{\alpha\beta\gamma}\Gamma^0_{\alpha\beta} B_\gamma , \quad (11)$$

$$\rho_B \equiv -\Gamma^\alpha_{\beta\alpha} B^\beta + \varepsilon^{\alpha\beta\gamma}\Gamma^0_{\alpha\beta} E_\gamma , \quad (12)$$

$$\begin{aligned}&\vec{j}_-\{E\} \equiv [-(\Gamma^\alpha_{0\beta} - \Gamma^\alpha_{\beta 0})E^\beta + \Gamma^\beta_{0\beta}E^\alpha \\ &- \varepsilon^{\alpha\beta\gamma}(\Gamma^0_{\beta 0}B_\gamma + \Gamma^\delta_{\beta\gamma}B_\delta)]\vec{e}_-\{\alpha\}\end{aligned} \quad (13)$$

$$\begin{aligned}&\vec{j}_-\{B\} \equiv [-(\Gamma^\alpha_{0\beta} - \Gamma^\alpha_{\beta 0})B^\beta + \Gamma^\beta_{0\beta}B^\alpha \\ &+ \varepsilon^{\alpha\beta\gamma}(\Gamma^0_{\beta 0}E_\gamma + \Gamma^\delta_{\beta\gamma}E_\delta)]\vec{e}_-\{\alpha\}\end{aligned} , \quad (14)$$

$$\Delta n \equiv -\gamma n(\Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{00}v_\alpha + \Gamma^\alpha_{\beta\alpha}v^\beta) , \quad (15)$$

$$g \equiv -\gamma\left[\Gamma^\alpha_{00} + (\Gamma^\alpha_{0\beta} + \Gamma^\alpha_{\beta 0})v^\beta + \Gamma^\alpha_{\beta\gamma}v^\beta v^\gamma\right]\vec{e}_\alpha , \quad (16)$$

where Γ^a_{bc} is the Ricci rotation coefficients associated with the tetrad $\{\vec{e}_\alpha\}$.

In the high frequency approximation the gravitational wave can be taken to be in the transverse and traceless (TT) gauge even in the presence of matter. In this gauge the metric of a linearized gravitational wave propagating in the z-direction is given by [8]

$$\begin{aligned}&ds^2 = -dt^2 + (1 + h_+)dx^2 + (1 - h_+)dy^2 \\ &+ 2h_\times dxdy + dz^2 ,\end{aligned} \quad (17)$$

where $h_+ \equiv h_+(z,t)$ and $h_\times \equiv h_\times(z,t)$ denote the two polarization modes of weak gravitational waves in TT-gauge, and $|h_+|, |h_\times| << 1$. The tetrad basis for the metric (17) is

$$\vec{e}_0 = \partial_t, \quad \vec{e}_1 = (1 - \tfrac{1}{2}h_+)\partial_x - \tfrac{1}{2}h_\times\partial_y ,$$

$$\vec{e}_2 = (1 + \tfrac{1}{2}h_+)\partial_y - \tfrac{1}{2}h_\times\partial_x , \quad \vec{e}_3 = \partial_z . \quad (18)$$

Using Eqs. (17) and (18) in the linearized field equation

$$\delta G_{ab} = k\delta T_{ab} \quad (19)$$

and subtracting the 11 and 22 and adding the 12 and 21 components of the equations, one gets

$$(\partial_t^2 - \partial_z^2)h_+ = k(\delta T_{11} - \delta T_{22}) \quad (20)$$

$$(\partial_t^2 - \partial_z^2)h_\times = k(\delta T_{12} + \delta T_{21}) . \quad (21)$$

In the basis (18) the non-zero terms in (11)-(16) are

$$\begin{aligned}&\vec{j}_E = -\frac{1}{2}(E_1\partial_t h_+ + B_2\partial_z h_+ + E_2\partial_t h_\times - B_1\partial_z h_\times)\vec{e}_1 \\ &+ \frac{1}{2}(E_2\partial_t h_+ - B_1\partial_z h_+ - E_1\partial_t h_\times - B_2\partial_z h_\times)\vec{e}_2 ,\end{aligned} \quad (22)$$

$$\begin{aligned}&\vec{j}_B = -\frac{1}{2}(B_1\partial_t h_+ - E_2\partial_z h_+ + B_2\partial_t h_\times + E_1\partial_z h_\times)\vec{e}_1 \\ &+ \frac{1}{2}(B_2\partial_t h_+ + E_1\partial_z h_+ - B_1\partial_t h_\times + E_2\partial_z h_\times)\vec{e}_2 ,\end{aligned} \quad (23)$$

$$\begin{aligned}&\vec{g} = -\frac{1}{2}\gamma(v_1\partial_t h_+ + v_1 v_3\partial_z h_+ + v_2\partial_t h_\times + v_2 v_3\partial_z h_\times)\vec{e}_1 \\ &+ \frac{1}{2}\gamma(v_2\partial_t h_+ + v_2 v_3\partial_z h_+ - v_1\partial_t h_\times - v_1 v_3\partial_z h_\times)\vec{e}_2 \\ &+ \frac{1}{2}\gamma((v_1^2 - v_2^2)\partial_z h_+ + 2v_1 v_2\partial_z h_\times)\vec{e}_3 .\end{aligned} \quad (24)$$

Here, propagation of gravitational wave perpendicular to the ambient magnetic field is considered. Therefore, it is assumed that $\vec{B}_0 = B_0\vec{e}_1$ and the perturbations are: $\vec{B} = (B_0 + B_x)\vec{e}_1$, $\vec{E} = E_y\vec{e}_2 + E_z\vec{e}_3$, $\vec{v} = v_y\vec{e}_2 + v_z\vec{e}_3$. It is noted that in the case of gravitational waves propagating in a plasma with an ambient magnetic field perpendicular to the direction of propagation, only h_+ - polarization part of the gravitational wave couples effectively to the electromagnetic wave. Therefore, for the simplicity it may be considered $h_\times \approx 0$. With these assumptions and approximations the Maxwell and fluid equations (5) - (10) reduce to the following set of equations:

$$\partial_z E_z = \sum_s q\gamma(n_0 + \delta n) \ , \qquad (25)$$

$$\partial_t E_y - \partial_z B_x = -\sum_s (q\gamma(n_0+\delta n)v_y) - \frac{1}{2}E_y\partial_t h_+ + \frac{1}{2}(B_0+B_x)\partial_z h_+ , \qquad (26)$$

$$\partial_t E_z = -\sum_s q\gamma(n_0 + \delta n)v_z \ , \qquad (27)$$

$$\partial_t B_x - \partial_z E_y = -\frac{1}{2}E_y\partial_z h_+ + \frac{1}{2}(B_0 + B_x)\partial_t h_+ , \qquad (28)$$

$$\partial_t(\gamma(n_0 + \delta n)) = -\partial_z(\gamma(n_0 + \delta n)v_z) , \qquad (29)$$

$$\partial_t(\gamma v_y) + v_z\partial_z(\gamma v_y) = \frac{q}{m}(E_y + v_z(B_0 + B_x)) , \qquad (30)$$

$$\partial_t(\gamma v_z) + v_z\partial_z(\gamma v_z) = \frac{q}{m}(E_z - v_y(B_0 + B_x)) , \qquad (31)$$

and linearized Einstein equation (20) becomes

$$(\partial_t^2 - \partial_z^2)h_+ = k(E_y^2 - 2B_0B_x - B_x^2) \ . \qquad (32)$$

Astronomical sources of gravitational waves might include radiation from spinning neutron stars, binary star systems, stellar gravitational collapse, stellar-mass and massive black holes [9]. For most binary astrophysical sources such as the binary neutron stars, the gravitational wave frequency is ~ 10^3 Hz [10]. Observations indicate that pulsars have enormous magnetic fields, typically in the range of $10^8 - 10^{12} G$, even there is some evidence that the fields in neutron stars may be as large $10^{14} G$ [11].

So under approximation $|2B_0B_x| >> |E_y^2 - B_x^2|$, and linearizing the system of equations (25) –(31) and from Eq. (32) we find

$$(\partial_t^2 - \partial_z^2)h_+ = -2kB_0B_x , \qquad (34)$$

$$\partial_t B_x - \partial_z E_y = \frac{1}{2}B_0\partial_t h_+ \ , \qquad (35)$$

$$\partial_t E_y - \partial_z B_x = -\sum_s q\gamma n_0 v_y + \frac{1}{2}B_0\partial_z h_+ , \qquad (36)$$

$$\partial_t(\gamma v_y) = \frac{q}{m}(E_y + v_z B_0) , \qquad (37)$$

$$\partial_t(\gamma v_z) = \frac{q}{m}(E_z - v_y B_0) , \qquad (38)$$

$$\partial_t E_z = -\sum_s q\gamma n_0 v_z \ . \qquad (39)$$

It is noted earlier that gravitational wave $h = h_+(z-t)$ interacts effectively with the electromagnetic waves only near the resonant conditions. Therefore, all the perturbations may be represented as $B_x = B_x(z-t)$, $E_y = E_y(z-t)$ and $\partial_t \approx -\partial_z$. For non-relativistic fluid velocity $\gamma \sim 1$. Then from Eq. (35), we get

$$B_x = -E_y + \frac{B_0}{2}h_+ \ . \qquad (40)$$

Using Eq.(40) in Eq. (34) and from Eq. (36) using Eqs. (35) and (37), one finds

$$(\partial_t^2 - \partial_z^2 + kB_0^2)h_+ = 2kB_0E_y \qquad (41)$$

$$(\partial_t^2 - \partial_z^2 + \omega_p^2)E_y = B_0\partial_z\partial_t h_+ \ , \qquad (42)$$

where $\omega_p = \sqrt{n_0 q^2 / m}$ is the plasma frequency and the slow plasma response is neglected. Equations (41) and (42) represent the coupled gravitational and electromagnetic waves in the strongly magnetized pulsar plasma.

3. Dark Solitons

In the super strong pulsar plasma we consider $\partial_t << \omega_c$; $\omega_c = qB_0 / mc$ (here, we put c to introduce dimensional quantities). In this case, Eq. (37) using Eq. (40) gives

$$v_z = -qE_y / m\omega_c = B_x / B_0 - h_+ / 2 , \qquad (43)$$

which is independent of charges of plasma particles. Therefore, the current along z -direction $j_z = 0$. Then $\partial_t E_z = -\sum qn_0 v_z = 0$ and neglecting the charge separation of electron-positron pulsar plasma, we put $E_z = 0$. Again, considering the high frequency response of v_y, we set $< v_y >= 0$ in the slow plasma dynamics. Then from continuity equation (29) and equation of motion (31) along $z-$direction, we find the following set of equations

$$\partial_t N + \partial_z v_z = 0 \ , \tag{44}$$

$$\partial_t v_z + \frac{1}{2}\partial_z |v_z|^2 = 0 \ , \tag{45}$$

which describes the nonlinear dynamics of gravitational wave-plasma interaction. Here, gravitational wave h_+ is incorporated through v_z [Eq.(43)]and $N = \delta n / n_0$. The above set of equations may be rewritten as

$$\partial_t^2 N - \partial_z^2 (\frac{q^2 |E_y|^2}{2m^2 \omega_c^2}) = 0 \quad . \tag{46}$$

For effective interaction $\partial_t^2 \approx \partial_z^2$, therefore, we may write

$$N = \frac{q^2}{2m^2 \omega_c^2} |E_y|^2 \ , \tag{47}$$

considering $|E_y|^2 \to 0$ at $z \to \pm\infty$.

Again, in this case $B_x \approx 0$ and $h_+ \approx 2 / B_0 E_y$. Therefore the wave equation (42) can be written as

$$(\partial_t^2 - c^2 \partial_z^2 + \omega_p^2) E_y = 2c \partial_z \partial_t E_y \quad . \tag{48}$$

Considering $E_y = \widetilde{E}_y(z,t) \exp[i(Kz - \omega t)]$, from Eq. (48), we find the dispersion relation

$$\omega^2 - K^2 c^2 - \omega_p^2 = -2Kc\omega \quad . \tag{49}$$

The right hand side of Eq. (49) represents the gravitational effect. We calculate the group velocity and group dispersion which are as follows:

$$v_g = \frac{d\omega}{dK} = c \left[\frac{2}{\sqrt{2 + \frac{\omega_p^2}{K^2 c^2}}} - 1 \right] \ , \tag{50}$$

$$v_g{}' = \frac{dv_g}{dK} = \frac{2\omega_p^2}{K^3 c (2 + \frac{\omega_p^2}{K^2 c^2})^{3/2}} . \tag{51}$$

We are looking for the slow dynamics of the emission envelope. Therefore, we move to a frame which moves at the group velocity v_g of the emission wave. Thus, we define the moving coordinates: $\zeta = z - v_g t$, $\tau = t$, Furthermore, we neglect $\partial^2{}_{\tau^2}, \partial_{\tau\xi}{}^2 -> 0$ considering the slow time variation of the plasma parameters.

The nonlinear interaction between electric emission and plasma perturbations produces an electric field envelope, which obeys (Karpman and Washimi [12]) the nonlinear Schrödinger equation (NLSE)

$$i(\partial_\tau + v_g \partial_\zeta)\widetilde{E}_y + \frac{1}{2} v_g' \partial_\zeta^2 \widetilde{E}_y - Q |\widetilde{E}_y|^2 \widetilde{E}_y = 0 \ , \tag{52}$$

where

$$Q = \frac{q^2 v_g c}{4m^2 V_A^2 (\omega - Kc)} \quad . \tag{53}$$

We seek the solution of Eq. (52) in the form

$$\widetilde{E}_y = E_y(\zeta, \tau) \exp[i\Psi(\zeta, \tau)]$$

with $\partial_\zeta \Psi = \kappa = const$ and $\partial_\tau \Psi = -\varpi = const$. Then the solution of the NLSE can be written as (Mofiz [13,14] ; Mofiz and Ferdous [15])

$$|E_y| = E_0 \tanh \left[\left| \frac{Q}{2v_g'} \right|^{1/2} E_0 \zeta \right] \ , \tag{54}$$

where the pulse amplitude

$$E_0 = \left| \frac{A}{Q} \right|^{1/2} , \tag{55}$$

with $A = \kappa v_g - \varpi - \frac{1}{2} v_g' \kappa^2$

and the pulse width

$$\delta = \left| \frac{2 v_g{}'}{Q E_0{}^2} \right|^{1/2} \quad . \tag{56}$$

It is to be noted that

$$\delta \cdot E_0 = \left| \frac{2v_g^{/}}{Q} \right|^{1/2} = \text{const.} \qquad (57)$$

The dimensionless amplitude of the soliton is given by

$$\overline{E}_0 = \frac{qE_0}{m\omega} = \left| \frac{4\kappa V_A^2(\omega - Kc)}{\omega^2 c} \right|^{1/2} \qquad (58)$$

which for superluminous wave $(\omega / K >> c)$ may be significant in the case of superstrong magnetic field of the pulsar where $V_A / c \sim 1$, where V_A is the Alfve'n velocity.

4. Conclusion

Based on Einstein–Maxwell system of MHD equations, we have studied the nonlinear propagation of a gravitational wave propagating perpendicular to a superstrong magnetic field immersed in an electron-positron pulsar plasma. It is found that a weak gravitational wave with h_+ polarization may resonate an effective field perturbation in the strongly magnetized plasma in an astrophysical context. From the continuity equation and the equation motion along the propagation, we find the density perturbation caused by the field fluctuation in the plasma. From the wave equation, we find the dispersion relation from which the group velocity and the group dispersion are determined. Density perturbation and field intensity variation in the plasma lead to a nonlinear frequency shift and the slowly varying field amplitude obeys the nonlinear Schrödinger equation (NLSE). The solution of the equation is the dark soliton, the amplitude of which may be very significant in the case of a superluminal wave propagation in the superstrong magnetic field of the pulsar plasma. The Goldreich-Julian charge density near the pulsar surface is in the order of $6.9 \times 10^{16} m^{-3}$ [16]. Gravitational radiation can generate large density perturbations, even for very small amplitudes, provided that the cyclotron frequency is much larger than the plasma frequency . Marklund *et. al.* [17] find a region where the magnetic field is static. In their case, the electromagnetic excitation due to the gravitational wave starts at $60R_s$ (R_s is the Schwarzschild radius). Therefore dark solitons formed by gravitational wave in this region may be localized which might have some astrophysical significances.

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