Fusion Modeling in Vlasov-Like Models

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The methods developed by the authors are applied to a few important reductions of BBGKY hierarchy, namely, various examples of Vlasov-like systems important both for fusion modeling and for particular physical problems related to plasma/beam physics. As in companion paper [1] devoted to the general situation of kinetic hierarchies, we concentrate mostly on the phenomena of localization and pattern formation to create a realistic model for fusion state in plasma.

Keywords: fusion, non-equilibrium ensmles, kinetic equations, multiresolution, localization

1. Vlasov-Poisson System. Description

In this paper we consider the applications of our approach based on variational multiresolution technique [1,2] to the systems with collective type behaviour described by some forms of Vlasov-Poisson/Maxwell equations, some important reduction of general BBGKY hierarchy [3,4]. It based on general ideology presented in companion paper [1], devoted to the general case of kinetic hierarchies. Such approach may be useful in all models in which it is possible and reasonable to reduce all complicated problems related to statistical distributions to the problems described by the systems of (non)linear ordinary/partial differential/integral equations with or without some (functional) constraints. E.g., in periodic accelerators and transport systems at the high beam currents and charge densities the effects of the intense self-fields, which are produced by the beam space charge and currents, determinine (possible) equilibrium states, stability and transport properties according to underlying nonlinear dynamics. The dynamics of such space-charge dominated high brightness beam systems can provide the understanding of the instability phenomena such as emittance growth, mismatch, halo formation related to the complicated behaviour of underlying hidden nonlinear modes outside of perturbative tori-like KAM regions [4]. Our analysis based on the variational-wavelet approach allows to consider polynomial and rational type of nonlinearities. In some sense in this particular case this approach is direct generalization of traditional nonlinear δF approach [4] in which weighted Klimontovich representation

$$\delta f_j = a_j \sum_{i=1}^{N_j} w_{ji}\delta(x - x_{ji})\delta(p - p_{ji}) \tag{1}$$

or self-similar decompostion like

$$\delta n_j = b_j \sum_{i=1}^{N_j} w_{ji}s(x - x_{ji}), \tag{2}$$

where $s(x - x_{ji})$ is a shape function of distributing particles on the grids in configuration space, are replaced by powerful technique of local nonlinear harmonic analysis, based on representation theory of underlying symmetries of functional space such as affine or more general. The solution has the multiscale/multiresolution decomposition via nonlinear high-localized eigenmodes, which corresponds to the full multiresolution expansion in all underlying time/phase space scales. Starting from Vlasov-Poisson equations, we consider the approach based on multiscale variational-wavelet formulation. We give the explicit representation for all dynamical variables in the base of compactly supported wavelets or nonlinear eigenmodes. Our solutions are parametrized by solutions of a number of reduced algebraical problems, one from which is nonlinear with the same degree of nonlinearity as initial problem and the others are the linear problems which correspond to the particular method of calculations inside concrete wavelet scheme. Because our approach started from variational formulation we can control evolution of instability on the pure algebraical level of reduced algebraical system of equations. This helps to control stability/unstability scenario of evolution in parameter space on pure algebraical level. In all these models numerical modeling demonstrates the appearance of coherent high-localized structures and as a result the (meta)stable patterns formation or unstable chaotic behaviour. Analysis based on the non-linear Vlasov equations leads to more clear understanding of collective effects and nonlinear beam dynamics of high intensity beam propagation in periodic-focusing and uniform-focusing transport systems. We consider the following form of equations [4]:

$$\begin{cases} \dfrac{\partial}{\partial s} + p_x\dfrac{\partial}{\partial x} + p_y\dfrac{\partial}{\partial y} - \left[k_x(s)x + \dfrac{\partial\psi}{\partial x}\right]\dfrac{\partial}{\partial p_x} - \\[2mm] \left[k_y(s)y + \dfrac{\partial\psi}{\partial y}\right]\dfrac{\partial}{\partial p_y}\Big\}f_b(x, y, p_x, p_y, s) = 0, \\[2mm] \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)\psi = -\dfrac{2\pi K_b}{N_b}\int \mathrm{d}p_x\mathrm{d}p_y f_b, \\[2mm] \int \mathrm{d}x\mathrm{d}y\mathrm{d}p_x\mathrm{d}p_y f_b = N_b. \end{cases} \tag{3}$$

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The corresponding Hamiltonian for transverse single-particle motion is given by

$$H(x, y, p_x, p_y, s) = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}[k_x(s)x^2$$
$$+ k_y(s)y^2] + H_1(x, y, p_x, p_y, s) + \psi(x, y, s), \qquad (4)$$

where H_1 is nonlinear (polynomial/rational) part of the full Hamiltonian and corresponding characteristic equations are:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} + k_x(s)x + \frac{\partial}{\partial x}\psi(x, y, s) = 0,$$
$$\frac{\mathrm{d}^2 y}{\mathrm{d}s^2} + k_y(s)y + \frac{\partial}{\partial y}\psi(x, y, s) = 0. \qquad (5)$$

2. Multiscale Representation

We obtain our multiscale/multiresolution representations for solutions of these equations via variational-wavelet approach. We decompose the solutions as

$$f_b(s, x, y, p_x, p_y) = \sum_{i=i_c}^{\infty} \oplus \delta^i f(s, x, y, p_x, p_y),$$

$$\psi(s, x, y) = \sum_{j=j_c}^{\infty} \oplus \delta^j \psi(s, x, y),$$

$$x(s) = \sum_{k=k_c}^{\infty} \oplus \delta^k x(s), \quad y(s) = \sum_{\ell=\ell_c}^{\infty} \oplus \delta^\ell y(s), \qquad (6)$$

where set

$$(i_c, j_c, k_c, \ell_c) \qquad (7)$$

corresponds to the coarsest level of resolution c in the full multiresolution decomposition [5]

$$V_c \subset V_{c+1} \subset V_{c+2} \subset \dots \qquad (8)$$

Introducing detail space W_j as the orthonormal complement of V_j with respect to

$$V_{j+1} : V_{j+1} = V_j \bigoplus W_j, \qquad (9)$$

we have for

$$f, \psi, x, y \subset L^2(\mathbf{R}) \qquad (10)$$

$$L^2(\mathbf{R}) = \overline{V_c \bigoplus_{j=c}^{\infty} W_j}. \qquad (11)$$

In some sense it is some generalization of the old δF approach [4]. Let L be an arbitrary (non) linear differential/integral operator with matrix dimension d, which acts on some set of functions

$$\Psi \equiv \Psi(s, x) = \left(\Psi^1(s, x), \dots, \Psi^d(s, x)\right),$$
$$s, x \in \Omega \subset \mathbf{R}^{n+1} \qquad (12)$$

from $L^2(\Omega)$:

$$L\Psi \equiv L(R(s, x), s, x)\Psi(s, x) = 0, \qquad (13)$$

where x are the generalized space coordinates or phase space coordinates, and s is "time" coordinate. After some anzatzes [2] the main reduced problem may be formulated as the system of ordinary differential equations

$$Q_i(f)\frac{\mathrm{d}f_i}{\mathrm{d}s} = P_i(f, s); \quad f = (f_1, \dots, f_n), \qquad (14)$$
$$i = 1, \dots, n; \ \max_i \deg P_i = p; \ \max_i \deg Q_i = q$$

or a set of such systems corresponding to each independent coordinate in phase space. They have the fixed initial (or boundary) conditions $f_i(0)$, where P_i, Q_i are not more than polynomial functions of dynamical variables f_j and have arbitrary dependence on time. As result we have the following reduced algebraical system of equations on the set of unknown coefficients λ_i^k of localized eigenmode expansion:

$$L(Q_{ij}, \lambda, \alpha_I) = M(P_{ij}, \lambda, \beta_J), \qquad (15)$$

where operators L and M are algebraization of RHS and LHS of initial problem and are unknowns of reduced system of algebraical equations (RSAE). After solution of RSAE (15) we determine the coefficients of wavelet expansion and therefore obtain the solution of our initial problem. It should be noted that if we consider only truncated expansion with N terms then we have the system of $N \times n$ algebraical equations with degree

$$\ell = max\{p, q\} \qquad (16)$$

and the degree of this algebraical system coincides with degree of initial differential system. So, we have the solution of the initial nonlinear (rational) problem in the form

$$f_i(s) = f_i(0) + \sum_{k=1}^{N} \lambda_i^k f_k(s), \qquad (17)$$

where coefficients λ_i^k are the roots of the corresponding reduced algebraical (polynomial) problem RSAE. Consequently, we have a parametrization of solution of initial problem by the solution of reduced algebraical problem. The obtained solutions are given in this form, where $f_k(t)$ are basis functions obtained via multiresolution expansions and represented by some compactly supported wavelets. As a result the solution of equations has the following multiscale/multiresolution decomposition via nonlinear high-localized eigenmodes, which corresponds to the full multiresolution expansion in all underlying scales starting from coarsest one. For

$$\mathbf{x} = (x, y, p_x, p_y) \qquad (18)$$

we will have

$$\Psi(s, \mathbf{x}) = \sum_{(i,j) \in Z^2} a_{ij} \mathbf{U}^i \otimes V^j(s, \mathbf{x}), \qquad (19)$$

$$V^j(s) = V_N^{j,slow}(s) + \sum_{l \geq N} V_l^j(\omega_l s), \quad \omega_l \sim 2^l$$

$$\mathbf{U}^i(\mathbf{x}) = \mathbf{U}_M^{i,slow}(\mathbf{x}) + \sum_{m \geq M} \mathbf{U}_m^i(k_m \mathbf{x}), \quad k_m \sim 2^m,$$

These formulae give us expansion into the slow part $\Psi_{N,M}^{slow}$ and fast oscillating parts for arbitrary N, M. So, we may move from coarse scales of resolution to the finest one for obtaining more detailed information about our dynamical process. The first terms in the RHS correspond on the global level of function space decomposition to resolution space and the second ones to detail space. It should be noted that such representations give the best possible localization properties in the corresponding (phase)space/time coordinates. In contrast with other approaches this formulae do not use perturbation technique or linearization procedures. So, by using wavelet bases with their good (phase) space/time localization properties we can describe high-localized (coherent) structures in spatially-extended stochastic systems with collective behaviour. Modeling demonstrates the appearance of (meta)stable patterns formation from high-localized coherent structures (fundamental eigenmodes, Fig. 1) or chaotic behaviour. On Fig. 2 we present contribution to the full expansion from coarsest level of decomposition. Fig. 3 shows the representations for full solutions, constructed from the first six scales (dilations) and demonstrates (meta) stable localized pattern formation in comparison with chaotic-like behaviour (Fig. 4) outside of KAM region. We can control the type of behaviour on the level of reduced algebraical system (15) which plays the role of Generalized Dispersion Relations (GDR).

3. Rate/RMS Models. Description

In this part we consider the applications of our technique based on the methods of local nonlinear harmonic analysis to nonlinear rms/rate equations for averaged quantities related to some particular case of nonlinear Vlasov–Maxwell equations [4]. We consider electrostatic approximation for a thin beam. This approximation is a particular important case of the general reduction from statistical collective description based on Vlasov–Maxwell equations to a finite number of ordinary differential equations for the second moments related quantities (beam radius and emittance). In our case these reduced rms/rate equations also contain some distribution averaged quantities besides the second moments, e.g. self-field energy of the beam particles. Such model is very efficient for analysis of many problems related to periodic focusing accelerators, e.g. heavy ion fusion and tritium production. So, we are interested in the understanding of collective properties, nonlinear dynamics and transport processes of intense non-neutral beams propagating through a periodic focusing field. Our approach allows to consider rational type of nonlinearities in rms/rate dynamical equations containing statistically averaged quantities also. The solution has the multiscale/multiresolution decomposition via nonlinear high-localized eigenmodes (Fig. 1), which corresponds to the full multiresolution expansion in all underlying internal hidden scales. We may move from coarse scales of

resolution to the finest one to obtain more detailed information about our dynamical process. In this way we give contribution to our full solution from each scale of resolution or each time/space scale or from each nonlinear eigenmode. Starting from some electrostatic approximation of Vlasov-Maxwell system and rms/rate dynamical models we consider the approach based on variational-wavelet formulation. We give explicit representation for all dynamical variables in the bases of compactly supported wavelets or nonlinear eigenmodes. Our solutions are parametrized by the solutions of a number of reduced standard algebraical problems like GDR mentioned above.

4. Rate Equations

In thin-beam approximation with negligibly small spread in axial momentum for beam particles we have in Larmor frame the following electrostatic approximation for Vlasov–Maxwell equations [4]:

$$\frac{\partial F}{\partial s} + x' \frac{\partial F}{\partial x} + y' \frac{\partial F}{\partial y} - \left(k(s)x + \frac{\partial \psi}{\partial x}\right)\frac{\partial F}{\partial x'}$$
$$-\left(k(s)y + \frac{\partial \psi}{\partial y}\right)\frac{\partial F}{\partial y'} = 0, \qquad (20)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = -\frac{2\pi K}{N}\int \mathrm{d}x'\mathrm{d}y'F, \qquad (21)$$

where $\psi(x, y, s)$ is normalized electrostatic potential and $F(x, y, x', y', s)$ is distribution function in transverse phase space (x, y, x', y', s) with normalization

$$N = \int \mathrm{d}x\mathrm{d}yn, \qquad n(x, y, s) = \int \mathrm{d}x'\mathrm{d}y'F, \qquad (22)$$

where K is self-field perveance which measures self-field intensity. Introducing self-field energy

$$E(s) = \frac{1}{4\pi K}\int \mathrm{d}x\mathrm{d}y|\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2| \qquad (23)$$

we have obvious equations for root-mean-square beam radius $R(s)$

$$R(s) =< x^2 + y^2 >^{1/2} \qquad (24)$$

and unnormalized beam emittance

$$\varepsilon^2(s) = 4(< x'^2 + y'^2 >< x^2 + y^2 > - < xx' - yy' >), \qquad (25)$$

which appear after averaging second-moments quantities regarding distribution function F [4]:

$$\frac{\mathrm{d}^2R(s)}{\mathrm{d}s^2} + \left(k(s)R(s) - \frac{K(1 + \Delta)}{2R^2(s)}\right)R(s) = \frac{\varepsilon^2(s)}{4R^3(s)}$$
$$\frac{\mathrm{d}\varepsilon^2(s)}{\mathrm{d}s} + 8R^2(s)\left(\frac{\mathrm{d}R}{\mathrm{d}s}\frac{K(1 + \Delta)}{2R(s)} - \frac{\mathrm{d}E(s)}{\mathrm{d}s}\right) = 0, \quad (26)$$

where the term

$$K(1 + \Delta)/2 \qquad (27)$$

may be fixed in some interesting cases, but generally we have it only as average

$$K(1 + \Delta)/2 = - < x\partial\psi/\partial x + y\partial\psi/\partial y > \qquad (28)$$

w.r.t. distribution F. Anyway, the rate equations represent reasonable reductions for the second-moments related quantities from the full nonlinear Vlasov-Poisson system. For trivial postulated distributions Davidson e.a. [4] found additional reductions. For KV distribution (step–function density) the second rate equation is trivial,

$$\varepsilon(s) = \text{const} \qquad (29)$$

and we have only one nontrivial rate equation for rms beam radius. The fixed-shape density profile ansatz for axisymmetric distributions also leads to similar situation: emittance conservation and the same envelope equation with two shifted constants only.

5. Multiscale Representation

Accordingly to our approach which allows us to find exact solutions as for Vlasov- like systems as for rms-like systems we need not to fix particular case of distribution function $F(x, y, x', y', s)$. Our consideration is based on the following multiscale N-mode ansatz:

$$F^N(x, y, x', y', s) =$$
$$\sum_{i_1,...,i_5=1}^{N} a_{i_1,...,i_5} \bigotimes_{k=1}^{5} A_{i_k}(x, y, x', y', s), \qquad (30)$$

$$\psi^N(x, y, s) = \sum_{j_1,j_2,j_3=1}^{N} b_{j_1,j_2,j_3} \bigotimes_{k=1}^{3} B_{j_k}(x, y, s). \qquad (31)$$

These formulae provide multiresolution representation for variational solutions of our system. Each high-localized mode/harmonics $A_j(s)$ corresponds to level j of resolution from the whole underlying infinite scale of spaces:

$$\dots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \dots, \qquad (32)$$

where the closed subspace $V_j (j \in \mathbf{Z})$ corresponds to level j of resolution, or to scale j. The construction of tensor algebra based on the multiscale bases [5] is considered in [2]. We will consider rate equations as the following operator equation. Let L, P, Q be an arbitrary nonlinear (rational in dynamical variables) first-order matrix differential operators with matrix dimension ($d = 4$ in our case) corresponding to the system of equations, which act on some set of functions

$$\Psi \equiv \Psi(s) = \left(\Psi^1(s), \dots, \Psi^d(s)\right), \quad s \in \Omega \subset R \qquad (33)$$

from $L^2(\Omega)$:

$$Q(R, s)\Psi(s) = P(R, s)\Psi(s) \qquad (34)$$

or

$$L\Psi \equiv L(R, s)\Psi(s) = 0 \qquad (35)$$

where

$$R \equiv R(s, \partial/\partial s, \Psi). \qquad (36)$$

Let us consider now the N mode approximation for solution as the following expansion in some high-localized wavelet-like basis:

$$\Psi^N(s) = \sum_{r=1}^{N} a_r^N \phi_r(s). \qquad (37)$$

We will determine the coefficients of expansion from the following variational condition:

$$L_k^N \equiv \int (L\Psi^N)\phi_k(s)\mathrm{d}s = 0 \qquad (38)$$

We have exactly dN algebraical equations for dN unknowns a_r. So, variational approach reduced the initial problem to the problem of solution of functional equations at the first stage and some algebraical problems at the second stage. As a result we have the following reduced algebraical system of equations (RSAE) on the set of unknown coefficients a_i^N of the expansion:

$$H(Q_{ij}, a_i^N, \alpha_I) = M(P_{ij}, a_i^N, \beta_J), \qquad (39)$$

where operators H and M are algebraization of RHS and LHS of initial problem. Q_{ij} (P_{ij}) are the coefficients of LHS (RHS) of the initial system of differential equations and as consequence are coefficients of RSAE (39).

$$I = (i_1, ..., i_{q+2}), \qquad J = (j_1, ..., j_{p+1}) \qquad (40)$$

are multiindices, by means of which are labelled α_I and β_I, the other coefficients of RSAE:

$$\beta_J = \{\beta_{j_1...j_{p+1}}\} = \int \prod_{1 \le j_k \le p+1} \phi_{j_k}, \qquad (41)$$

where p is the degree of polynomial operator

$$\alpha_I = \{\alpha_{i_1}...\alpha_{i_{q+2}}\} = \sum_{i_1,...,i_{q+2}} \int \phi_{i_1}...\dot\phi_{i_s}...\phi_{i_{q+2}}, \qquad (42)$$

where q is the degree of polynomial operator Q,

$$i_\ell = (1, ..., q + 2), \quad \dot\phi_{i_s} = \mathrm{d}\phi_{i_s}/\mathrm{d}s. \qquad (43)$$

We may extend our approach to the case when we have additional constraints on the set of our dynamical variables

$$\Psi = \{R, \varepsilon\} \qquad (44)$$

and additional averaged terms also. In this case by using the method of Lagrangian multipliers we may again apply the same approach but for the extended set of variables. As a result we obtain the expanded system of algebraical equations analogous to our system. Then, after reduction we can again extract from its solution the coefficients of the expansion. It should be noted that if we consider only truncated expansion with N terms then we have the system of $N \times d$ algebraical equations with the degree

$$\ell = max\{p, q\} \qquad (45)$$

and the degree of this algebraical system coincides with the degree of the initial system. So, after all we have the solution of the initial nonlinear (rational) problem in the form

$$R^N(s) = R(0) + \sum_{k=1}^{N} a_k^N \phi_k(s),$$

$$\varepsilon^N(s) = \varepsilon(0) + \sum_{k=1}^{N} b_k^N \phi_k(s), \qquad (46)$$

where coefficients a_k^N, b_k^N are the roots of the corresponding reduced algebraical (polynomial) problem RSAE. Consequently, we have a parametrization of the solution of the initial problem by solution of reduced algebraical problem. The problem of computations of coefficients α_I, β_J of reduced algebraical system may be explicitly solved in wavelet framework via various approaches [5]. The obtained solutions are given in the form, where $\phi_k(s)$ are proper wavelet bases functions (e.g., periodic or boundary). It should be noted that such representations give the best possible localization properties in the corresponding (phase)space/time coordinates. In contrast with different approaches these formulae do not use perturbation technique or linearization procedures and represent dynamics via generalized nonlinear localized eigenmodes expansion. Our N mode construction gives the following general multiscale representation:

$$R(s) = R_N^{slow}(s) + \sum_{i \geq N} R^i(\omega_i s), \quad \omega_i \sim 2^i,$$

$$\varepsilon(s) = \varepsilon_N^{slow}(s) + \sum_{j \geq N} \varepsilon^j(\omega_j s), \quad \omega_j \sim 2^j, \quad (47)$$

where $R^i(s), \varepsilon^j(s)$ are represented by some family of (nonlinear) eigenmodes and gives the full multiresolution/multiscale representation in the high-localized wavelet bases. As a result we can construct various unstable (Fig. 4) or (meta) stable (non-gaussian!) patterns (Fig. 3, Fig. 7) from high-localized (coherent) fundamental modes (Fig. 1) in complicated stochastic systems with complex collective behaviour. Definitely, partition(s) as generic dynamical variable(s) cannot be postulated (e.g., as some sort of gaussians or something else fixed apriori) but need to be computed as solutions of proper stochastic dynamical evolution model. Only after that, it is possible to calculate other dynamical quantities and physically interesting averages.

6. Towards Energy Confinement

Analysis and modeling considered in this and companion paper [1] describes, in principle, a scenario for the generation of controllable localized (meta) stable fusion-like state (Fig.3 and Fig.7). Definitely, chaotic-like unstable partitions/states (Fig. 4) dominate during non-equilibrium evolution. It means that (possible) localized (meta) stable partitions have measure equal to zero a.e. on the full space of hierarchy of partitions defined on a domain of the definition in the whole phase space. Nevertheless, our Generalized Dispersion Relations, like (15) or (39), give some chance to build the controllable localized state (Fig. 7) starting from initial chaotic-like partition (Fig. 4) via process of controllable self-organization. Figures 5 and 6 demonstrate two subsequent steps towards creation important fusion or confinement state, Fig. 7, which can be characterized by the presence of a few important physical modes only in contrast with the opposite, chaotic-like state, described by infinite number of contributed modes. Of course, such confinement states, characterized by zero measure and minimum entropy (we name them waveletons), can be only metastable. But these long-living fluctuations can and must be very important from the practical point of view, because the averaged time of existence of such states may be even more than needed for practical realization, e.g., in controllable fusion processes. Further details will be considered elsewhere.
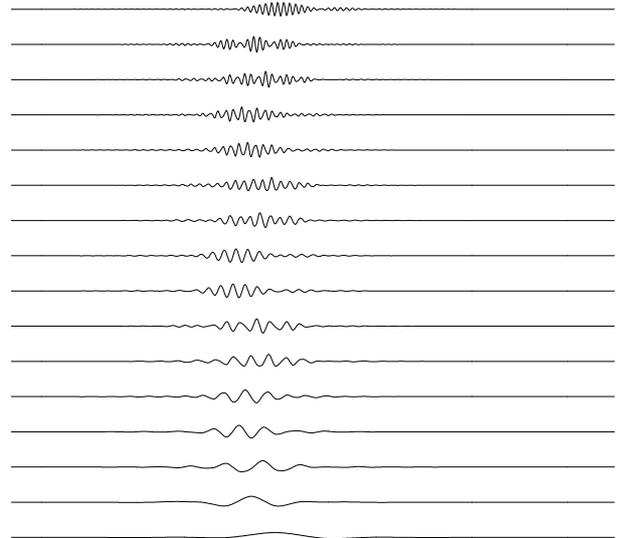
Fig. 1   Localized modes.

[1] A.N. Fedorova and M.G. Zeitlin, Thorny Path to Fusion in Plasma: Fusion State as a Waveleton, This Volume.

[2] A.N. Fedorova and M.G. Zeitlin, a full list of papers/preprints (including links to arXiv) can be found on web sites above.

[3] A. H. Boozer, Rev. Mod. Phys., 76, 1071 (2004).

[4] R.C. Davidson and H. Qin, Physics of Intense Charged Particle Beams in High Energy Accelerators (World Scientific, Singapore, 2001); A.W. Chao, Physics of Collective Beam Instabilities in High Energy Accelerators (Wiley, New York, 1993); R. Balescu, Equilibrium and Nonequilibrium Statistical Mechanics, (Wiley, New York, 1975).

[5] Y. Meyer, Wavelets and Operators (Cambridge Univ. Press, 1990); D. Donoho, WaveLab (Stanford, 2000).
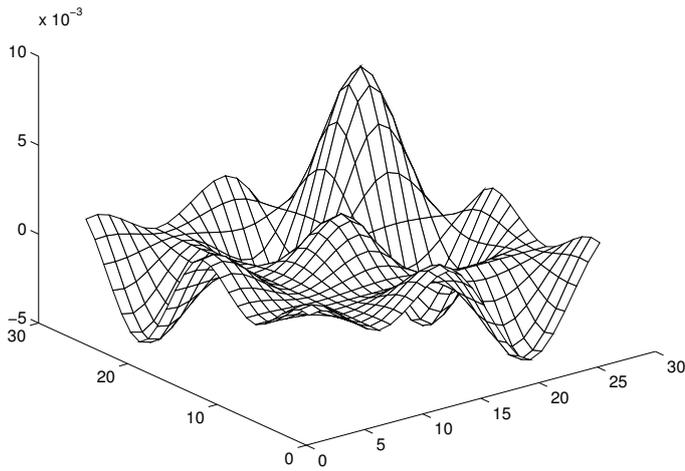
Fig. 2   Localized two-dimensional partition.
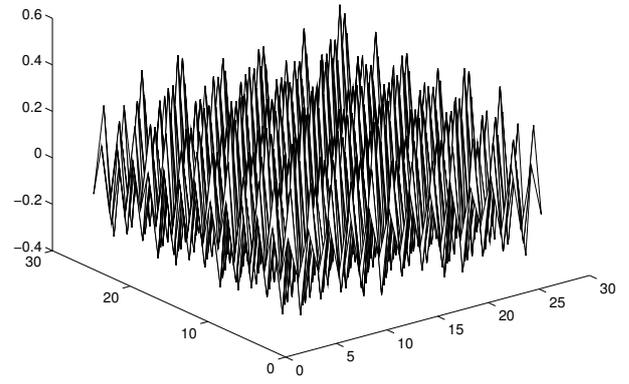
Fig. 5   Self-organization, step 1: towards confinement.
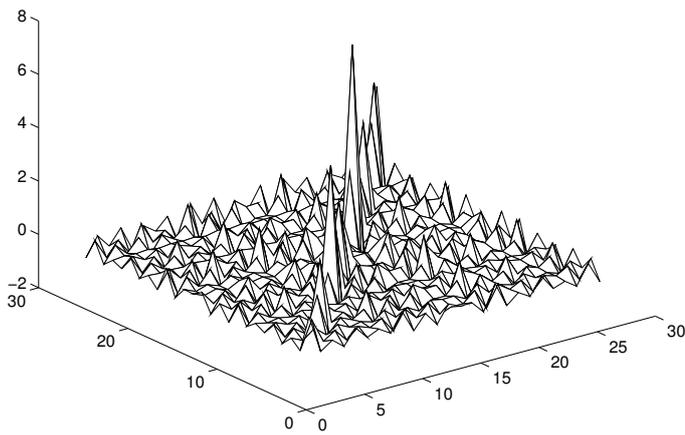
Fig. 3   Metastable pattern.
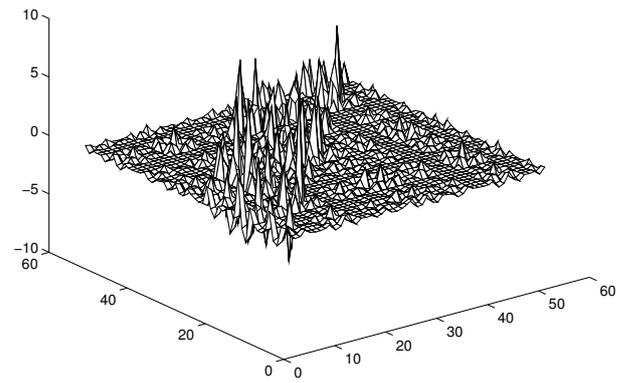
Fig. 6   Self-organization, step 2: towards confinement.
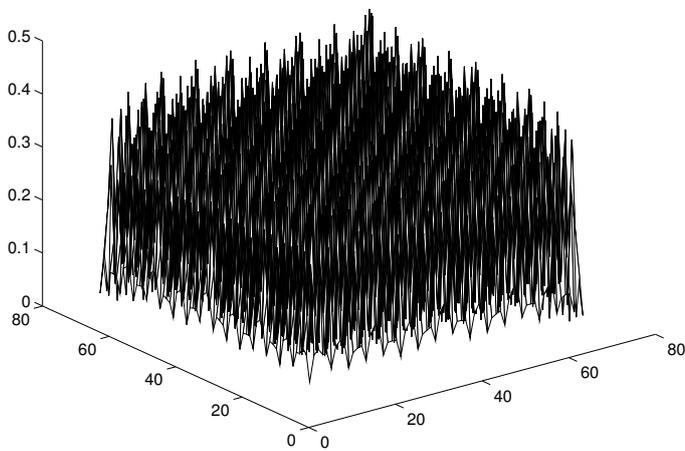
Fig. 4   Chaotic pattern.
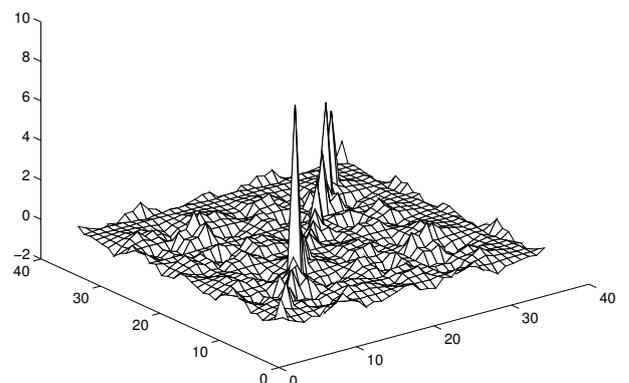
Fig. 7   Localized pattern(waveleton): energy confinement state.