

Improved Formalism for Flowing Two-Fluid Equilibrium and Its Application to ST

ISHIDA Akio, HARAHAHAP Caesar O, STEINHAUER Loren C.¹ and PENG Martin Y-K.²

Niigata University, Niigata 950-2181, Japan

¹*University of Washington, Redmond Plasma Physics Laboratory, Redmond, WA USA*

²*Oak Ridge National Laboratory, Oak Ridge, TN USA*

(Received: 9 December 2003 / Accepted: 24 March 2004)

Abstract

An improved formalism for flowing two-fluid equilibria with non-uniform density is developed in the limit of no dissipation. The system for axisymmetric equilibria can be expressed using second order differential equations for the stream functions of the generalized vorticities of the electron and ion fluids plus an algebraic equation for the density. These simultaneous equations have six arbitrary functions for the stream functions of poloidal flows, the specific entropies and the generalized enthalpies for the electron and ion fluids. It is found that the type of the equation for the ion generalized vorticity is *elliptic* (*hyperbolic*) when the poloidal ion flow speed is less (larger) than the sonic speed. This property is different from the single-fluid model. As significant poloidal ion flow may exist even in the present experiment, this difference is getting important.

Keywords:

flowing equilibrium, two-fluid model, transition condition from elliptic to hyperbolic

1. Introduction

It has been shown that two-fluid effects are important in many cases such as stability of a high beta plasma [1], relaxation of a flowing high beta plasma [2,3] and equilibrium of a flowing plasma [4]. In particular it was found that the two-fluid effect can be significant depending on the effective size of gradient of physical quantities, the beta value and how close the flow speed is to the ion diamagnetic drift speed [4]. Unfortunately because of the complexity of this system, computational modeling to date has assumed the density is uniform and has paid attention mainly to the effect of flow. The formalism to analyze a flowing two-fluid equilibrium with non-uniform density in the limit of no dissipation was developed in [5]. Recently we found an improved way to treat the entropies for the electron and ion fluids. The equation for axisymmetric equilibria can be expressed using second order differential equations for the stream functions of the generalized vorticities of the electron and ion fluids plus an algebraic Bernoulli equation for the density. These simultaneous equations have six arbitrary functions for the stream functions of poloidal flows, the specific entropies and the generalized enthalpies for the electron and ion fluids. In this paper an improved formalism for flowing two-fluid equilibria with non-uniform density is presented. We also give a discussion of the changes in the class of the governing equations as the ion poloidal flow speed increases. This property is quite dif-

ferent from the single-fluid model [6].

2. Basic equations

Throughout this section, we are concerned with a general multi-fluid. Its equilibrium is governed by the following system of equations.

$$\nabla \cdot (n_\alpha \mathbf{u}_\alpha) = 0 \quad (2.1)$$

$$m_\alpha (\mathbf{u}_\alpha \cdot \nabla) \mathbf{u}_\alpha = \frac{\nabla p_\alpha}{n_\alpha} + q_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + \frac{\mathbf{R}_\alpha}{n_\alpha} \quad (2.2)$$

$$\frac{1}{\gamma - 1} n_\alpha \mathbf{u}_\alpha \cdot \nabla T_\alpha + p_\alpha \nabla \cdot \mathbf{u}_\alpha = -\nabla \cdot \mathbf{Q}_\alpha \quad (2.3)$$

$$p_\alpha = n_\alpha T_\alpha \quad (2.4)$$

$$\nabla \times \mathbf{B} = \mu_0 \sum_\alpha q_\alpha n_\alpha \mathbf{u}_\alpha \quad (2.5)$$

where α and γ denote the species and the adiabatic constant, respectively, and T_α is the temperature in the energy unit. \mathbf{R}_α and \mathbf{Q}_α represent the visco-resistive force and the heat flux, respectively which will be neglected in the equilibrium analysis of Sec.3.

Introduce, for each species, the specific entropy s_α (per particle) as

$$s_\alpha = \frac{1}{\gamma - 1} \ln \left(\frac{C_D T_\alpha^\gamma}{p_\alpha^{\gamma-1}} \right) \quad (2.6)$$

where C_D is a dimensional constant. By utilizing (2.6), (2.1) and (2.4), the equation of energy transfer (2.3) can be written as

$$p_\alpha(\mathbf{u}_\alpha \cdot \nabla)s_\alpha = -\nabla \cdot \mathbf{Q}_\alpha \quad (2.7)$$

Since this equation shows that the specific entropy s_α is dimensionless, C_D has a dimension of $[C_D] = [n^{\gamma-1}T^{-1}]$. The equation (2.6) shows that the temperature or the enthalpy can be expressed in terms of p_α and s_α . Thus, the specific enthalpy (per particle) h_α can be written as a function of p_α and s_α , which is the canonical form of the species enthalpy,

$$h_\alpha(p_\alpha, s_\alpha) = \frac{\gamma}{\gamma-1} C_D^{-\frac{1}{\gamma}} p_\alpha^{\frac{\gamma-1}{\gamma}} \exp\left(\frac{\gamma-1}{\gamma} s_\alpha\right) \quad (2.8)$$

Note that from (2.8) the familiar thermodynamic relations follow:

$$\frac{\partial h_\alpha}{\partial p_\alpha} = \frac{1}{n_\alpha} \quad \text{and} \quad \frac{\partial h_\alpha}{\partial s_\alpha} = T_\alpha$$

For later convenience, we write here the expressions for h_α and T_α in terms of s_α and n_α ,

$$h_\alpha = \frac{1}{\gamma-1} T_\alpha \quad (2.9)$$

$$T_\alpha = \frac{1}{C_D} n_\alpha^{\gamma-1} \exp((\gamma-1)s_\alpha) \quad (2.10)$$

We now return to the equation of motion (2.2). Introduce the generalized momentum \mathbf{P}_α and its vorticity $\mathbf{\Omega}_\alpha$ as in Ref.3

$$\mathbf{P}_\alpha \equiv m_\alpha \mathbf{u}_\alpha + q_\alpha \mathbf{A}, \quad (2.11)$$

$$\mathbf{\Omega}_\alpha \equiv \nabla \times \mathbf{P}_\alpha \quad (2.12)$$

where \mathbf{A} is the vector potential, i.e. $\mathbf{B} = \nabla \times \mathbf{A}$. Then, using the vector identity $(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha = \nabla(u_\alpha^2/2) - \mathbf{u}_\alpha \times \nabla \times \mathbf{u}_\alpha$ (2.2) becomes

$$-\mathbf{u}_\alpha \times \mathbf{\Omega}_\alpha + \nabla H_\alpha - T_\alpha \nabla s_\alpha = \frac{\mathbf{R}_\alpha}{n_\alpha} \quad (2.13)$$

where the generalized enthalpy H_α is defined by

$$H_\alpha \equiv h_\alpha + \frac{1}{2} m_\alpha u_\alpha^2 + q_\alpha V_E, \quad (2.14)$$

V_E is the electrostatic potential, i.e. $\mathbf{E} = -\nabla V_E$ and the following relation is used.

$$\frac{\nabla p_\alpha}{n_\alpha} = \nabla h_\alpha - T_\alpha \nabla s_\alpha$$

3. Axisymmetric flowing two-fluid equilibrium

Henceforth, we limit attention to equilibria of a two-fluid composed of hydrogen-like ions and electrons and assume that charge quasi-neutrality ($n_i \approx n_e = n$) holds. To reach equilibrium, the dissipation is necessary. The thermodynamics, however, shows that there are many cases that the dissipation itself is not important to describe some equilibri-

um properties. So we assume that the visco-resistive force \mathbf{R}_α as well as the heat conduction $\nabla \cdot \mathbf{Q}_\alpha$ are negligible in the equilibrium. Then, the equilibrium equations are from (2.1), (2.5), (2.7) and (2.13),

$$\nabla \cdot (n\mathbf{u}_\alpha) = 0 \quad (3.1)$$

$$\mathbf{u}_\alpha \times \mathbf{\Omega}_\alpha = \nabla H_\alpha - T_\alpha \nabla s_\alpha \quad (3.2)$$

$$\mathbf{u}_\alpha \cdot \nabla s_\alpha = 0 \quad (3.3)$$

$$\nabla \times \mathbf{B} = \mu_0 en(\mathbf{u}_i - \mathbf{u}_e) \quad (3.4)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (3.5)$$

We adopt here cylindrical coordinates (R, ϕ, Z) and restrict our attention to axisymmetric equilibrium where any physical quantity does not depend on the coordinate ϕ . The divergence-free quantities such as \mathbf{B} , $n\mathbf{u}_\alpha$, and $\mathbf{\Omega}_\alpha$ can then be expressed using their stream functions as

$$\mathbf{B} = \nabla \psi_m(R, Z) \times \nabla \phi + \mathbf{B}_\phi(R, Z) \hat{\phi} \quad (3.6)$$

$$n\mathbf{u}_\alpha = \nabla \psi_\alpha(R, Z) \times \nabla \phi + n(R, Z) u_{\alpha\phi}(R, Z) \hat{\phi} \quad (3.7)$$

$$\mathbf{\Omega}_\alpha = q_\alpha \nabla \psi_\alpha(R, Z) \times \nabla \phi + \mathbf{\Omega}_{\alpha\phi}(R, Z) \hat{\phi} \quad (3.8)$$

where $\psi_m, \psi_i, \psi_e, \Psi_i, \Psi_e$ are the desired stream functions and $\hat{\phi}$ is the unit vector in the toroidal direction. A fact that there is no toroidal component of (3.2) implies that ψ_α is an arbitrary function of Ψ_α , the principal surface function for species α , i.e.

$$\psi_\alpha = \psi_\alpha(\Psi_\alpha) \quad (3.9)$$

Applying (3.9) into (3.3) shows that s_α is an arbitrary function of Ψ_α i.e.

$$s_\alpha = s_\alpha(\Psi_\alpha) \quad (3.10)$$

As seen from (2.9) and (2.10), it is more convenient to introduce a new function $w_\alpha(\Psi_\alpha)$ defined by

$$w_\alpha(\Psi_\alpha) \equiv \exp[(\gamma-1)s_\alpha(\Psi_\alpha)]. \quad (3.11)$$

Note that $w_\alpha(\Psi_\alpha)$ has a statistical meaning; it is the ‘‘number of microscopic state per particle’’ recalling Boltzmann’s law $S = k \ln W$. In terms of it the temperature is written as

$$T_\alpha = \frac{1}{C_D} n^{\gamma-1} w_\alpha(\Psi_\alpha) \quad (3.12)$$

Taking the inner product of (3.2) with $\mathbf{\Omega}_\alpha$ shows that H_α is an arbitrary function of Ψ_α i.e.

$$H_\alpha = H_\alpha(\Psi_\alpha) \quad (3.13)$$

In the following treat the electron as mass-less. To aid numerical computation a set of dimensionless variables is defined. These are based on the system length scale R_r , the reference magnetic field B_r and the reference density n_r . These give the reference Alfvén speed $V_A = B_r / \sqrt{\mu_0 m_i n_r}$ as in [4]. Then the closed system of equations governing axisymmetric flowing two-fluid equilibria is given by three equa-

tions. The first two arise from the transverse (to Ω_α) components of the equations of motion for electrons and ions:

$$\begin{aligned} 2x \frac{\partial^2 \Psi_e}{\partial x^2} + \frac{\partial^2 \Psi_e}{\partial Z^2} &= S_*^2 \psi_e'(\Psi_e) \{ \psi_i(\Psi_i) \\ &- \psi_e(\Psi_e) \} - S_*^2 n (\Psi_i - \Psi_e) \\ &- 2xn \left[H_e'(\Psi_e) - \frac{1}{\gamma-1} n^{\gamma-1} w_e'(\Psi_e) \right] \end{aligned} \quad (3.14)$$

where $S_* = R_r \sqrt{\mu_0 e^2 n_r / m_i}$, $x = R^2/2$ is new radial coordinate and the prime denotes differentiation with respect to Ψ_α .

$$\begin{aligned} &\frac{(\psi_i'(\Psi_i))^2}{n} \left\{ 2x \frac{\partial^2 \Psi_i}{\partial x^2} + \frac{\partial^2 \Psi_i}{\partial Z^2} - \left[\left(\frac{1}{n} \frac{\partial n}{\partial x} \right) 2x \frac{\partial \Psi_i}{\partial x} + \left(\frac{1}{n} \frac{\partial n}{\partial Z} \right) \frac{\partial \Psi_i}{\partial Z} \right] \right\} \\ &= - \frac{\psi_i'(\Psi_i) \psi_i''(\Psi_i)}{n} \left\{ 2x \left(\frac{\partial \Psi_i}{\partial x} \right)^2 + \left(\frac{\partial \Psi_i}{\partial Z} \right)^2 \right\} \\ &+ S_*^2 \psi_i'(\Psi_i) [\psi_i(\Psi_i) - \psi_e(\Psi_e)] - S_*^2 n (\Psi_i - \Psi_e) \\ &+ 2xn \left[H_i'(\Psi_i) - \frac{1}{\gamma-1} n^{\gamma-1} w_i'(\Psi_i) \right] \end{aligned} \quad (3.15)$$

The third equation arises from the sum of the parallel (to Ω_α) components of the two equations of motion, i.e. a Bernoulli relation,

$$\begin{aligned} &\frac{\gamma}{\gamma-1} n^{\gamma-1} [w_e(\Psi_e) + w_i(\Psi_i)] - \left[H_e(\Psi_e) + H_i(\Psi_i) - \frac{S_*^2}{4x} (\Psi_i - \Psi_e)^2 \right] \\ &+ \frac{1}{2} \left[\left(\frac{\psi_i'(\Psi_i)}{n} \frac{\partial \Psi_i}{\partial x} \right)^2 + \frac{1}{2x} \left(\frac{\psi_i'(\Psi_i)}{n} \frac{\partial \Psi_i}{\partial Z} \right)^2 \right] = 0 \end{aligned} \quad (3.16)$$

Of course, any other physical quantity can be expressed in terms of the principal surface functions Ψ_e and Ψ_i . Note that the first term of the Bernoulli relation (3.16) is just sum of the enthalpies $h_e + h_i$ in dimensionless expression and the term proportional to $\Psi_i - \Psi_e$ results from the ion toroidal flow while the third term is due to the ion poloidal flow.

Examine the class of the differential eqs. (3.14) and (3.15) for the principal electron and ion surface variables Ψ_e and Ψ_i , respectively. The type of (3.14) which comes from Ampere's law is clearly *elliptic*. Noting that some of the second derivatives result from the gradients of the last term in (3.16), the second order derivatives in (3.15) can be written as

$$S_{xx} \frac{\partial^2 \Psi_i}{\partial x^2} + S_{xz} \frac{\partial^2 \Psi_i}{\partial x \partial Z} + S_{zz} \frac{\partial^2 \Psi_i}{\partial Z^2} \quad (3.17)$$

where

$$S_{zz} = 1 + M_Z^2 (1 - M_p^2)^{-1}, \quad S_{zz} = 2x S_{zz} \text{ and} \\ S_{zz} = -2\sqrt{2x} M_Z M_R (1 - M_p^2)^{-1}.$$

Here

$$\begin{aligned} M_R &= \frac{\psi_i'(\Psi_i)}{n} \frac{\partial \Psi_i}{\partial x} [\gamma n^{\gamma-1} (w_e(\Psi_e) + w_i(\Psi_i))]^{-1/2}, \\ M_Z &= \frac{-\psi_i'(\Psi_i)}{\sqrt{2xn}} \frac{\partial \Psi_i}{\partial Z} [\gamma n^{\gamma-1} (w_e(\Psi_e) + w_i(\Psi_i))]^{-1/2}, \end{aligned}$$

and $M_p^2 = M_R^2 + M_Z^2$. Noting the temperature $T_\alpha = n^{\gamma-1} w_\alpha(\Psi_\alpha)$, M_R and M_Z can be written as

$$M_R = u_{iR} [\gamma(T_e + T_i)]^{-1/2} \text{ and } M_Z = u_{iZ} [\gamma(T_e + T_i)]^{-1/2}.$$

These are the Mach numbers of the poloidal components of the ion flow. Thus the type of (3.15) is determined by the sign of the following quantity

$$S_{xx} S_{zz} - \left(\frac{S_{xz}}{2} \right)^2 = 2x(1 - M_p^2)^{-1}$$

When the poloidal Mach number is less than unity, $M_p < 1$ (i.e. $u_{ip} < \sqrt{\gamma(T_e + T_i)/m_i}$ in dimensional expression), the type of (3.15) is elliptic while when the poloidal Mach number is larger than unity, $M_p > 1$ (i.e. $u_{ip} > \sqrt{\gamma(T_e + T_i)/m_i}$ in dimensional expression), the type of (3.15) is *hyperbolic*. This property is different from that of the single-fluid model [6] in the following two points:

- (1) The above ratio does not depend on the magnitude of the poloidal magnetic field.
- (2) The ratio depends on the adiabatic constant γ .

4. Summary and discussion

An improved formalism for flowing two-fluid equilibrium with non-uniform density is developed. The governing equations are given by (3.14), (3.15) and (3.16). Equation (3.14) is always elliptic (assuming massless electrons) while eq.(3.15) is elliptic (hyperbolic) when the poloidal ion flow speed is less (larger) than the sonic speed. This property is quite different from the single-fluid model. As significant poloidal flow may exist in JT-60U [7], this difference will be important.

Presently we are solving the 1D case of the foregoing equilibrium equations including poloidal flow as well as toroidal flow to compare the NSTX equilibrium and to clarify the importance of two-fluid effects.

References

- [1] A. Ishida, H. Momota and L.C. Steinhauer, *Phys. Fluids* **31**, 3024 (1988).
- [2] L.C. Steinhauer and A. Ishida, *Phys. Rev. Lett.* **79**, 3423 (1997).
- [3] L.C. Steinhauer and A. Ishida, *Phys. Plasmas* **5**, 2609 (1998).
- [4] H. Yamada, T. Katano, K. Kanai, A. Ishida and L.C. Steinhauer, *Phys. Plasmas* **9**, 4605 (2002).
- [5] L.C. Steinhauer, *Phys. Plasmas* **6**, 2734 (1999).
- [6] E. Hameiri, *Phys. Fluids* **26**, 230 (1983).
- [7] *private communication with* Dr. T. Fujita, JAERI (2003).