J. Plasma Fusion Res. SERIES, Vol. 6 (2004) 210–213

A Model Equation for Ballooning Modes in Toroidally Rotating Tokamaks

FURUKAWA Masaru* and TOKUDA Shinji

Naka Fusion Research Establishment, Japan Atomic Energy Research Institute, Ibaraki, 311-0193, Japan

(Received: 9 December 2003 / Accepted: 11 May 2004)

Abstract

A model equation for ballooning modes in toroidally rotating tokamaks is derived. It is confirmed that the model equation is appropriate for analyzing the stabilization mechanism of the ballooning modes by comparing the numerical solutions of the model equation with those of the original ballooning equations.

Keywords:

toroidal rotation, ballooning mode, tokamak, magnetohydrodynamics

1. Introduction

In the H-mode [1] pedestal region, the reachable pressure gradient is often limited by magnetohydrodynamic (MHD) activities, called edge localized modes (ELMs) [2]. One of the causes of the ELMs is considered to be ideal (non-dissipative) MHD ballooning instabilities [2]. Thus the linear stability analysis of the ballooning mode [3,4] has been a crucial issue in tokamak fusion research.

Plasmas in the pedestal region of tokamaks often rotate. The plasma rotation affects linear stability of MHD ballooning modes. It was numerically found that toroidal rotation shear stabilizes ideal MHD high-n ballooning modes (n: toroidal mode number) [5-8]. We found that the toroidal rotation shear damps the perturbation energy of the ballooning modes, which results in the stabilization of the mode [8]. However, it has not been clarified how the toroidal rotation shear damps the perturbation energy.

An analytical study may clearly show the damping mechanism of perturbation energy, however, the ballooning equations [6] solved numerically in ref. [8] are too complicated to analyze; they are coupled wave equations for two components of the displacement vector. Here, we will derive a model equation which is a wave equation for one variable. We will also solve the model equation numerically, and compare the solutions with those of the original ballooning equations. It will be shown that the model equation is appropriate for analyzing the damping mechanism of the perturbation energy of ballooning modes.

In Sec. **2**, the model equation is derived by simplifying the original ballooning equations. In Sec. **3**, numerical solutions of the model equation are compared with those of the original ballooning equations. Summary is given in Sec. **4**.

2. Derivation

In this Section, we derive a model equation appropriate for analyzing the damping mechanism of the perturbation energy of ballooning modes.

It was numerically shown that toroidal rotation shear stabilizes ballooning modes even in the incompressible limit [5,7]. Thus, we adopt the incompressible model. The incompressible ballooning equations are obtained by taking the variation of the action with a constraint: the action is

$$I \equiv \int \mathrm{d}t \int \mathrm{d}^3x(\mathcal{T} - \mathcal{V} + \lambda \mathcal{W}),$$

where

$$\begin{aligned}
\mathcal{T} &\equiv \rho \left| \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} \right|^2, \\
\mathcal{V} &\equiv -\boldsymbol{\xi}^* \cdot \mathcal{F}(\boldsymbol{\xi}), \\
\mathcal{W} &\equiv \left| \nabla \cdot \boldsymbol{\xi} \right|^2, \\
\mathcal{F}(\boldsymbol{\xi}) &\equiv \frac{1}{\mu_0} \left[\left(\nabla \times \boldsymbol{Q} \right) \times \boldsymbol{B} + \left(\nabla \times \boldsymbol{B} \right) \times \boldsymbol{Q} \right] \\
&\quad + \nabla \left(\gamma p \nabla \cdot \boldsymbol{\xi} + \boldsymbol{\xi} \cdot \nabla p \right) + \nabla \cdot \left(\rho \boldsymbol{\xi} \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right), \\
\boldsymbol{Q} &\equiv \nabla \times \left(\boldsymbol{\xi} \times \boldsymbol{B} \right).
\end{aligned}$$

The incompressible condition is imposed by $\lambda \mathcal{W}$, where λ is a Lagrange multiplier. The displacement vector is $\boldsymbol{\xi}$, ρ and p

Corresponding author's e-mail: furukawm@fusion.naka.jaeri.go.jp

**Present address: Graduate School of Frontier Sciences, The University of Tokyo, Kashiwa, 277-8561, Japan*

e-mail: furukawa@ppl.k.u-tokyo.ac.jp

are equilibrium mass density and pressure, respectively, γ is a specific heat ratio, and μ_0 is a vacuum permeability. The equilibriu magnetic field and velocity are $\boldsymbol{B} = \chi'(\hat{\chi})\,(\nabla\zeta \times \nabla\hat{\chi} - q\nabla\hat{\chi} \times \nabla\theta)$ and $\boldsymbol{v} = \Omega(\hat{\chi})\boldsymbol{e}_\zeta$, respectively, where $\hat{\chi} \equiv (\chi - \chi_{axis})/(\chi_{edge} - \chi_{axis})$ is a normalized poloidal flux, χ is a poloidal flux function, the prime denotes the derivative with respect to $\hat{\chi}$, θ and ζ are poloidal and toroidal angle, respectively, q is a safety factor, Ω is a toroidal rotation frequency, $\boldsymbol{e}_\zeta$ is a covariant basis vector in the toroidal direction. The operator $\mathcal{F}$ is self-adjoint [9].

The displacement vector is expressed in the eikonal form and expanded by $1/n \ll 1$ as

$$\boldsymbol{\xi} = \left(\hat{\boldsymbol{\xi}}^{(0)}\left(\hat{\chi}, \vartheta, t\right) + \frac{1}{n}\,\hat{\boldsymbol{\xi}}^{(1)}\left(\hat{\chi}, \vartheta, t\right) + \cdots\right) \mathrm{e}^{\mathrm{i}nS(\hat{\chi}, \alpha, t)},$$

where S is the eikonal, n is a toroidal mode number, ϑ is an extended poloidal angle in the covering space, $\alpha \equiv \zeta - q\theta$ is a field line label. The eikonal satisfies

$$\boldsymbol{B} \cdot \nabla S = 0,$$
$$\frac{\mathrm{d}S}{\mathrm{d}t} = 0.$$

The wave vector is defined as $\hat{\boldsymbol{k}} \equiv \nabla S = \nabla\zeta - q\nabla\theta - (\vartheta - \theta_k + \dot{\Omega} t)\nabla q$, where θ_k is a ballooning angle and $\dot{\Omega} \equiv \mathrm{d}\Omega/\mathrm{d}q$. The displacement vector in the eikonal form, as well as $\lambda = \lambda^{(0)} + \lambda^{(1)}/n + \cdots$, are substituted in the action, then we obtain incompressible ballooning equations in $O(n^0)$;

$$\rho\left|\hat{\boldsymbol{k}}\right|^2 \frac{\partial^2 \hat{\xi}_\perp^{(0)}}{\partial t^2} + 2\rho\left(\hat{\boldsymbol{k}}\cdot\nabla\Omega\right)\frac{\partial \hat{\xi}_\perp^{(0)}}{\partial t} - 2\rho\Omega B^2\left(\hat{\boldsymbol{k}}\cdot\hat{Z}\right)\frac{\partial \hat{\xi}_\parallel^{(0)}}{\partial t}$$
$$= \frac{B^2}{\mu_0}\boldsymbol{B}\cdot\nabla\left(\frac{\left|\hat{\boldsymbol{k}}\right|^2}{B^2}\boldsymbol{B}\cdot\nabla\hat{\xi}_\perp^{(0)}\right) + \frac{2B^2}{\mu_0}\left(B\times\hat{\boldsymbol{k}}\cdot\boldsymbol{\kappa}\right)\boldsymbol{B}\cdot\nabla\hat{\xi}_\parallel^{(0)}$$
$$- \frac{\rho R\Omega^2}{p}\left(\boldsymbol{B}\cdot\nabla p\right)\left(B\times\hat{\boldsymbol{k}}\cdot\hat{R}\right)\hat{\xi}_\parallel^{(0)}$$
$$+ \left\{\left(\boldsymbol{B}\times\hat{\boldsymbol{k}}\right)\cdot\left[2\left.\frac{\partial p}{\partial\chi}\right|_R \boldsymbol{\kappa} - \left.\frac{\partial\left(\rho R^2\Omega^2\right)}{\partial\chi}\right|_R \frac{\hat{\boldsymbol{R}}}{R} + \frac{2\rho\Omega}{B^2}\frac{\mathrm{d}\Omega}{\mathrm{d}\chi}\left(\boldsymbol{B}\cdot\hat{Z}\right)\nabla\chi\right] - \frac{4}{\mu_0}\left(\boldsymbol{B}\times\hat{\boldsymbol{k}}\cdot\boldsymbol{\kappa}\right)^2 - \frac{\rho^2 R^2\Omega^4}{p\,B^2}\left(\boldsymbol{B}\times\hat{\boldsymbol{k}}\cdot\hat{R}\right)^2\right\}\hat{\xi}_\perp^{(0)}, \qquad (1)$$

$$\rho B^2\frac{\partial^2\hat{\xi}_\parallel^{(0)}}{\partial t^2} - 2\rho\Omega\left(\hat{\boldsymbol{k}}\cdot\hat{Z}\right)\frac{\partial\hat{\xi}_\perp^{(0)}}{\partial t} + 2\rho\Omega\left(\nabla\Omega\cdot\hat{Z}\right)\hat{\xi}_\perp^{(0)}$$
$$= \frac{1}{\mu_0}\boldsymbol{B}\cdot\nabla\left[B^2\boldsymbol{B}\cdot\nabla\hat{\xi}_\parallel^{(0)} - 2\left(\boldsymbol{B}\times\hat{\boldsymbol{k}}\cdot\boldsymbol{\kappa}\right)\hat{\xi}_\perp^{(0)}\right]$$
$$- \frac{\boldsymbol{B}\cdot\nabla p}{p}\left[\left(\boldsymbol{B}\cdot\nabla p\right)\hat{\xi}_\parallel^{(0)} + \frac{\rho R\Omega^2}{B^2}\left(\boldsymbol{B}\times\hat{\boldsymbol{k}}\cdot\hat{R}\right)\hat{\xi}_\perp^{(0)}\right]. \qquad (2)$$

Here, $\hat{\boldsymbol{\xi}}^{(0)} = \hat{\xi}_\perp^{(0)}\boldsymbol{B}\times\hat{\boldsymbol{k}}/B^2 + \hat{\xi}_\parallel^{(0)}\boldsymbol{B}$, $\hat{R}$ and $\hat{Z}$ are unit vectors in the R and Z direction of a cylindrical coordinate system, respectively, and $\boldsymbol{\kappa} \equiv (\boldsymbol{B}/B)\cdot\nabla(\boldsymbol{B}/B)$ is a magnetic curvature. Equations (1) and (2) are coupled wave equations for $\hat{\xi}_\perp^{(0)}$ and $\hat{\xi}_\parallel^{(0)}$ along the extended poloidal angle. Thus, the incompressibility does not simplify the ballooning equations significantly.

Next, we set $\Omega = 0$ in eqs. (1) and (2), since it was shown that the perturbation energy damps even for $\Omega = 0$ if $\Omega' \neq 0$ [7,8]. The resulting equations are

$$\mu_0\rho\left(\left|\hat{\boldsymbol{k}}\right|^2\frac{\partial^2\hat{\xi}_\perp^{(0)}}{\partial t^2} - 2\hat{\boldsymbol{k}}\cdot\nabla\Omega\frac{\partial\hat{\xi}_\perp^{(0)}}{\partial t}\right)$$
$$= B^2\boldsymbol{B}\cdot\nabla\left(\frac{\left|\hat{\boldsymbol{k}}\right|^2}{B^2}\boldsymbol{B}\cdot\nabla\hat{\xi}_\perp^{(0)}\right)$$
$$+ 2\left(\boldsymbol{B}\times\hat{\boldsymbol{k}}\cdot\boldsymbol{\kappa}\right)\left(\frac{\mu_0}{B^2}\boldsymbol{B}\times\hat{\boldsymbol{k}}\cdot\nabla p - 2\boldsymbol{B}\times\hat{\boldsymbol{k}}\cdot\boldsymbol{\kappa}\right)\hat{\xi}_\perp^{(0)}$$
$$+ 2B^2\left(\boldsymbol{B}\times\hat{\boldsymbol{k}}\cdot\boldsymbol{\kappa}\right)\boldsymbol{B}\cdot\nabla\hat{\xi}_\parallel^{(0)}, \qquad (3)$$

$$\mu_0\rho B^2\frac{\partial^2\hat{\xi}_\parallel^{(0)}}{\partial t^2} = \boldsymbol{B}\cdot\nabla\left[B^2\boldsymbol{B}\cdot\nabla\hat{\xi}_\parallel^{(0)} - 2\left(\boldsymbol{B}\times\hat{\boldsymbol{k}}\cdot\boldsymbol{\kappa}\right)\hat{\xi}_\perp^{(0)}\right]. \qquad (4)$$

Equations (3) and (4) are much simpler than eqs. (1) and (2), however, they are still coupled equations for $\hat{\xi}_\perp^{(0)}$ and $\hat{\xi}_\parallel^{(0)}$. This coupling does not come from the rotation shear. For a static plasma, we commonly examine the ballooning stability by solving a second-order differential equation for a perpendicular component of a ballooning displacement. This equation is derived from an incompressible part of potential energy and kinetic energy of only the perpendicular component of the displacement. When we take a variation of the potential energy and the kinetic energy of both the perpendicular and parallel components with imposing the incompressibility, we obtain coupled equations even for a static plasma. It is also noted that eqs. (3) and (4) coincide with the incompressible limit (the specific heat ratio γ is taken to be infinity) of eqs. (51) and (52) of ref. [4].

Here, we propose to decouple $\hat{\xi}_\parallel^{(0)}$ from eq. (3) by dropping the last two terms in that equation;

$$\mu_0\rho\left(\left|\hat{\boldsymbol{k}}\right|^2\frac{\partial^2\hat{\xi}_\perp^{(0)}}{\partial t^2} - 2\hat{\boldsymbol{k}}\cdot\nabla\Omega\frac{\partial\hat{\xi}_\perp^{(0)}}{\partial t}\right)$$
$$= B^2\boldsymbol{B}\cdot\nabla\left(\frac{\left|\hat{\boldsymbol{k}}\right|^2}{B^2}\boldsymbol{B}\cdot\nabla\hat{\xi}_\perp^{(0)}\right)$$
$$+ \frac{2\mu_0}{B^2}\left(\boldsymbol{B}\times\hat{\boldsymbol{k}}\cdot\boldsymbol{\kappa}\right)\left(\boldsymbol{B}\times\hat{\boldsymbol{k}}\cdot\nabla p\right)\hat{\xi}_\perp^{(0)}. \qquad (5)$$

The reason why we drop not only $2B^2(\boldsymbol{B}\times\hat{\boldsymbol{k}}\cdot\boldsymbol{\kappa})\boldsymbol{B}\cdot\nabla\hat{\xi}_\parallel^{(0)}$ but also $-4(\boldsymbol{B}\times\hat{\boldsymbol{k}}\cdot\boldsymbol{\kappa})^2\hat{\xi}_\perp^{(0)}$ is as follows. From eq. (4), the last two terms of eq. (3) express kinetic energy of $\hat{\xi}_{\parallel}^{(0)}$. This means that we drop kinetic energy of the parallel component of the displacement. As noted above, the ballooning equation for a static plasma is derived similarly. Actually, when $\Omega' = 0$, eq. (5) coincide with the ballooning equation for a static plasma. Therefore, eq. (5) nicely connects ballooning modes in a rotating plasma with those in a static plasma.

3. Numerical solutions

In this Section, we compare numerical solutions of the model equation (5) with those of the original ballooning equations including compressibility. At first, we must confirm that the toroidal rotation shear can stabilize the ballooning mode even by the model equation. Figure 1 shows the growth rate $\gamma\tau_A$ as a function of toroidal rotation shear $\Omega'\tau_A$, where τ_A is the Alfvén time. The magnetic shear parameter S and the pressure gradient parameter α are chosen as follows; (i) low magnetic shear case $S = 0.5$ and $\alpha = 0.7$ and (ii) high magnetic shear case $S = 2$ and $\alpha = 2$. The equilibrium is a large aspect ratio and circular cross-section tokamak. We found that the ballooning mode is stabilized by increasing the toroidal rotation shear in both cases. At several values of $\Omega'\tau_A$, the growth rate suddenly decreases. Such a behavior of the growth rate was also found in ref. [7].

Figure 2 shows the time evolution of the perturbation energy $\| \xi_\perp^2 \|$ for (a) $S = 0.5$, $\alpha = 0.7$ and $\Omega' = 0.05$, (b) $S = 0.5$, $\alpha = 0.7$ and $\Omega' = 0.2$, (c) $S = 2$, $\alpha = 2$ and $\Omega' = 0.1$ and (d) $S = 2$, $\alpha = 2$ and $\Omega' = 0.8$. In Figs. 2(a), 2(c) and 2(d), we found the periodic time evolution occurs with damping phases even by the model equation. Figure 2(b) shows the time evolution of $\| \xi_\perp^2 \|$ for a stable case. The perturbation energy

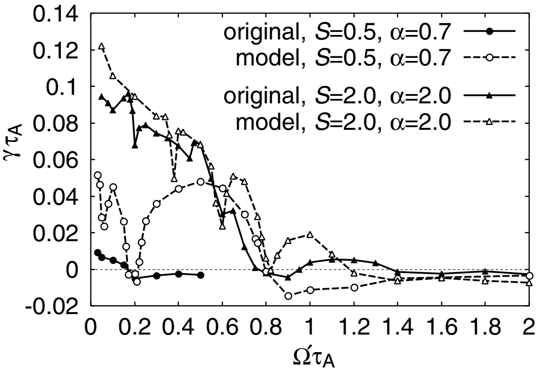

Fig. 1 Growth rate $\gamma\tau_A$ as a function of toroidal rotation shear $\Omega'\tau_A$. The ballooning mode is stabilized by increasing the toroidal rotation shear even by the model equation.

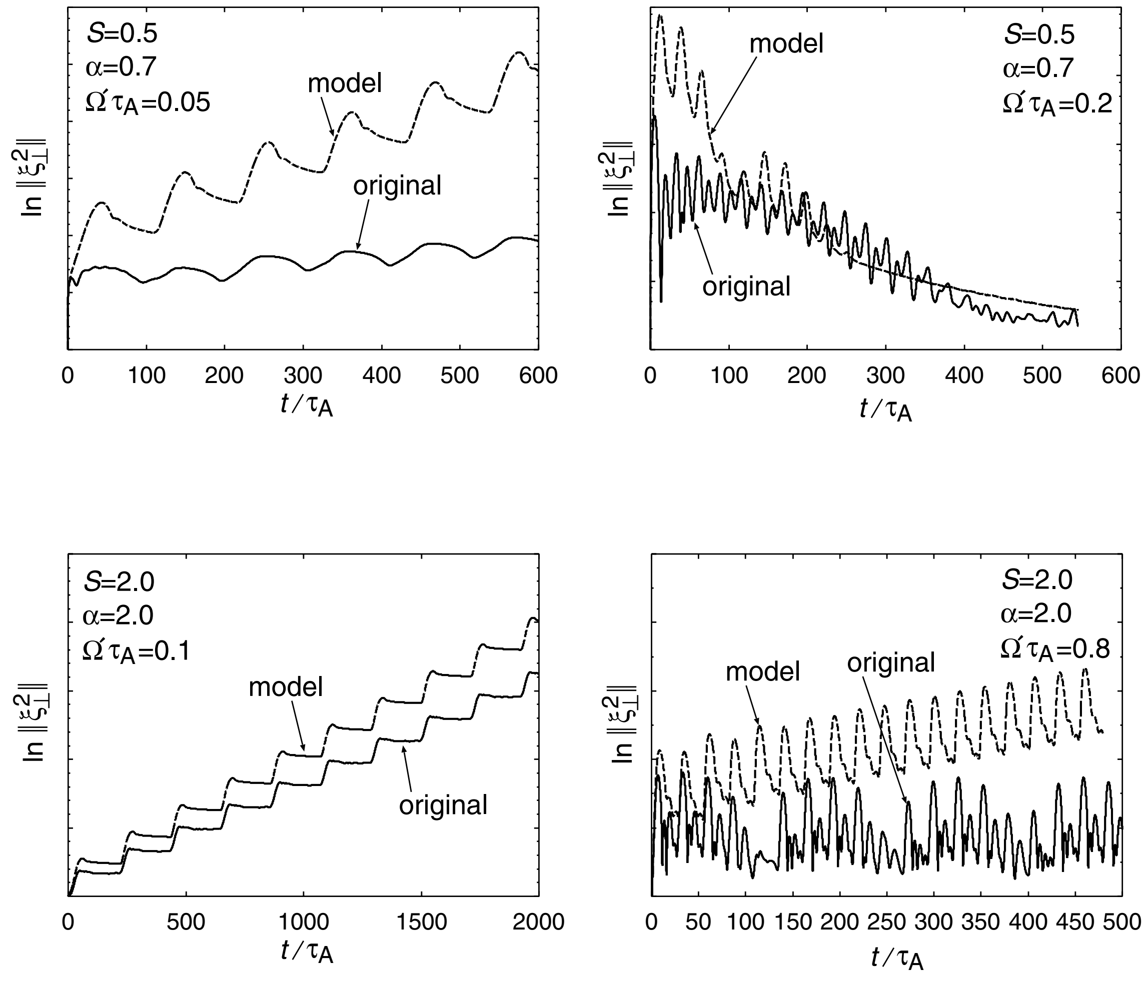

Fig. 2 Time evolution of perturbation energy $\| \xi_\perp^2 \|$. Periodic time evolution with damping phases occurs even by the model equation.

$\|\xi_\perp^2\|$ damps with oscillations even by the model equation.

4. Summary

We have derived a model equation for analyzing the mechanism of the damping of perturbation energy of ballooning modes by toroidal rotation shear. We have confirmed that the model equation is appropriate for analyzing the damping mechanism by comparing the numerical solutions with those of the original ballooning equations. We speculate that stabilization of ballooning modes by toroidal rotation shear originates from both (i) the time dependence of the coefficients of the model equation and (ii) the convection term. The mechanism of stabilization will be reported elsewhere.

Acknowledgements

The authors thank Dr. H. Ninomiya, Dr. M. Kikuchi and Dr. Y. Kishimoto for a discussion and comments.

References

[1] F. Wagner, G. Becker *et al.*, Phys. Rev. Lett. **49**, 1408 (1982).
[2] J.W. Connor, Plasma Phys. Control. Fusion **40**, 531 (1998).
[3] J.W. Connor, R.J. Hastie and J.B. Taylor, Phys. Rev. Lett. **40**, 396 (1978).
[4] R.L. Dewar and A.H. Glasser, Phys. Fluids **26**, 3038 (1983).
[5] W.A. Cooper, Plasma Phys. Control. Fusion **30**, 1805 (1988).
[6] F.L. Waelbroeck and L. Chen, Phys. Fluids B **3**, 601 (1991).
[7] R.L. Miller, F.L. Waelbroeck *et al.*, Phys. Plasmas **2**, 3676 (1995).
[8] M. Furukawa, S. Tokuda and M. Wakatani, Nucl. Fusion **43**, 425 (2003).
[9] A. Bhattacharjee, R. Iacono *et al.*, Phys. Fluids B **1**, 2207 (1989).