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# Theory of the Newcomb Equation and Applications to MHD Stability Analysis of a Tokamak

TOKUDA Shinji and AIBA Nobuyuki[1]
*Naka Fusion Research Establishment, Japan Atomic Energy Research Institute, Ibaraki, 311-0193, Japan*
*[1]The Graduate University for Advanced Studies, Toki, 509-5292, Japan*
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**Abstract**

Recent progress in the theory of the Newcomb equation is reported. Emphasis is put on the analysis of external modes including peeling modes (high $n$ kink modes), where $n$ is the toroidal mode number. A theory for low $n$ external modes is developed so that it is also useful for the analysis of resistive wall modes.

**Keywords:**

Newcomb equation, eigenvalue problem, ideal MHD stability, external mode, edge stability, tokamak

## 1. Introduction

It is well known that the Newcomb equation, the inertia free linear ideal magnetohydrodynamic (MHD) equation [1], plays fundamental roles in the MHD stability theory. A code MARG2D has been developed which solves numerically the 2-dimensional Newcomb equation and the associated eigenvalue problem [2]. In this paper, recent research on the Newcomb equation is reported. The main focus is to develop tools for the analysis of low $n$ and high $n$ external modes, where $n$ is the toroidal mode number. The high $n$ kink modes, called peeling modes [3], recently get attention in the study on MHD stability of tokamak edge plasmas [4]. For the low $n$ external modes, we develop a theory that expresses the change of potential energy due to the plasma displacement by a quadratic form with respect to the values of the displacement at the plasma surface. This formulation is useful for the analysis of resistive wall modes. For the analysis of peeling modes we extend the MARG2D formulation into the vacuum region by expressing the perturbation of magnetic fields in vacuum by a suitable vector potential.

## 2. Newcomb equation

In an axisymmetric toroidal system such as a tokamak, equilibrium magnetic fields are expressed as

$$\boldsymbol{B}_{eq} = \nabla\phi \times \nabla\psi + F\nabla\phi\,, \tag{1}$$

where the cylindrical coordinate system ($R$, $Z$, $\phi$) is employed; $\psi(R, Z)$ and $F(\psi)$ are, respectively, the poloidal flux function and toroidal field function. We define the radial coordinate by

$$r^2 := 2R_0\int_0^{\psi} \frac{q}{F}\mathrm{d}\psi. \tag{2}$$

Here, $R_0$ is the position of the magnetic axis (for simplicity, the mirror symmetry of the equilibrium is assumed: $\psi(R, -Z) = \psi(R, Z)$); $q(r)$ is the safety factor, as usual. We define the poloidal angle $\theta$ so that the magnetic field lines are straight; the Jacobian of the coordinate system ($r$, $\theta$, $\phi$) is $\sqrt{g} = rR^2/R_0$. Let $\vec{\zeta}$ be an infinitesimal plasma displacement with the toroidal mode number $n$ that is incompressible ($\nabla \cdot \vec{\zeta} = 0$), and let

$$X(r,\theta) := \vec{\zeta}\cdot\nabla r, \tag{3}$$

$$V(r,\theta) := r\left(\vec{\zeta}\cdot\nabla\theta - \frac{1}{q}\vec{\zeta}\cdot\nabla\phi\right), \tag{4}$$

then the change of the plasma potential energy $W_p$ due to the displacement $\vec{\zeta}$ reads [5]

$$W_p = \pi\int L\,\mathrm{d}r\mathrm{d}\theta, \tag{5}$$

and the Lagrangian density function $L$ reads

$$\begin{aligned} L \;=\; & a\,|D_\theta(X)|^2 + c\,|\partial_r(rX)+\partial_\theta V|^2 \\ & + b\,|inV + \frac{1}{q}\partial_r(rX) + hX + r\beta_{r\theta}D_\theta(X)|^2 \\ & + e\,|X|^2\,. \end{aligned} \tag{6}$$

Here, the operator $D_\theta(X)$ is defined by

$$D_\theta(X) := \frac{1}{q}\partial_\theta X - inX, \tag{7}$$

and the other coefficients are given in ref. [2].

By minimizing $W_p$ with respect to $V(r, \theta)$, we obtain the reduced energy integral

*Corresponding author's e-mail: tokudas@fusion.naka.jaeri.go.jp*

$$W_p[\boldsymbol{X},\boldsymbol{X}] = 2\pi^2 \int_0^a L[\boldsymbol{X},\boldsymbol{X}]\mathrm{d}r. \tag{8}$$

Here, the vector function $\boldsymbol{X}(r)$ is defined as

$$\boldsymbol{X}(r) := \{X_{-Lf}(r), \ldots, X_{Lf}(r)\}^t, \tag{9}$$

by using the poloidal Fourier harmonics $X_l(r)$

$$X(r,\theta) = \sum_{l=-L_f}^{L_f} X_l(r)\exp(\mathrm{i}l\theta), \tag{10}$$

where $L_f$ is the truncated poloidal mode number. And the reduced Lagrangian density is

$$L[\boldsymbol{X},\boldsymbol{X}] = \left\langle \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}r} | \boldsymbol{L} | \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}r} \right\rangle + \langle \boldsymbol{X} | \boldsymbol{K} | \boldsymbol{X} \rangle + \left\langle \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}r} | \boldsymbol{M}^t | \boldsymbol{X} \right\rangle + \left\langle \boldsymbol{X} | \boldsymbol{M} | \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}r} \right\rangle, \tag{11}$$

where $\boldsymbol{L}$, $\boldsymbol{M}$, $\boldsymbol{K}$ are matrices; $\boldsymbol{L}$ and $\boldsymbol{K}$ are hermitian, the details of which are given in ref. [2], and

$$\langle \boldsymbol{X} | \boldsymbol{K} | \boldsymbol{X} \rangle := \sum_{j,k} X_j K_{jk} X_k.$$

From $L[\boldsymbol{X}, \boldsymbol{X}]$, we have the 2-D Newcomb equation

$$\mathcal{N}\boldsymbol{X} := -\frac{\mathrm{d}}{\mathrm{d}r}\left(\boldsymbol{L}\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}r}\right) - \frac{\mathrm{d}}{\mathrm{d}r}(\boldsymbol{M}^t\boldsymbol{X}) + \boldsymbol{M}\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}r} + \boldsymbol{K}\boldsymbol{X} = 0. \tag{12}$$

The eigenvalue problem associated with eq. (12) is given by

$$\mathcal{N}\boldsymbol{X} = -\lambda R \boldsymbol{X}, \tag{13}$$

where $R$ is a multiplicative and diagonal operator whose components are $R_{m,m} \propto (m/q - n)^2$. The natural boundary condition for $X_m$ is imposed at the rational surface of $r_m$ ($m = nq(r_m)$); the continuous conditions are imposed for other harmonics $X_l$ ($l \neq m$). A code MARG2D which solves eqs. (12) and (13) has been developed by using a finite element method. The code has been applied to identify stable states for ideal MHD internal perturbations [2].

## 3. Application to the theory of external modes

The bilinear form associated with eq. (8) is given by

$$W[\vec{\xi}, \vec{\eta}] = W_p[\vec{\xi}, \vec{\eta}] + \langle \vec{\xi}_a | \boldsymbol{M}_V | \vec{\eta}_a \rangle, \tag{14}$$

where

$$\vec{\xi}_a = \vec{\xi}(a), \tag{15}$$

and the matrix $\boldsymbol{M}_V$ stands for the contribution from the vacuum region. Let $S = \{\vec{\xi} \,|\mathcal{N}\vec{\xi} = 0\}$ be a set of functions that satisfy the Newcomb equation. If $\vec{\xi}(r)$, $\vec{\eta}(r) \in S$, then we have

$$W_p[\vec{\xi},\vec{\eta}] = \langle \vec{\xi}_a | \boldsymbol{M}_H | \vec{\eta}_a \rangle + \frac{1}{2}\left\langle \vec{\xi}_a | \boldsymbol{L} | \frac{\mathrm{d}\vec{\eta}_a}{\mathrm{d}r} \right\rangle + \frac{1}{2}\left\langle \frac{\mathrm{d}\vec{\xi}_a}{\mathrm{d}r} | \boldsymbol{L} | \vec{\eta}_a \right\rangle, \tag{16}$$

$$\boldsymbol{M}_H := \frac{1}{2}(\boldsymbol{M} + \boldsymbol{M}^t). \tag{17}$$

Now let us make a vector function $\boldsymbol{Y}^m(r) \in S$

$$\boldsymbol{Y}^m(r) = (Y^m_{-L_f}(r), \ldots, Y^m_{L_f}(r))^t, \tag{18}$$

for $m = 0, \pm 1, \ldots, \pm L_f$, where each poloidal harmonics $Y_l^m(r)$ satisfies the condition

$$Y_l^m(a) = 0 \quad (l \neq m), \quad Y_m^m(a) = 1, \quad l = 0, \pm 1, \ldots, \pm L_f. \tag{19}$$

The set $\{\boldsymbol{Y}^m(r)\}$ forms a basis [6]. An external mode can be expressed by using an arbitrary set $\{x_m\}$ of real numbers as

$$\vec{\xi}(r) = \sum_m x_m \boldsymbol{Y}^m(r). \tag{20}$$

The change of the potential energy due to $\vec{\xi}$ is given by the quadratic form of the vector $\boldsymbol{x}$,

$$W[\vec{\xi}, \vec{\xi}] = \langle \boldsymbol{x} | \boldsymbol{A} | \boldsymbol{x} \rangle, \tag{21}$$

where the matrix $\boldsymbol{A}$, which is real and symmetric, is given by

$$\boldsymbol{A} = \boldsymbol{M}_p + \boldsymbol{M}_V, \tag{22}$$

$$M_{p(l,m)} = W_p[\boldsymbol{Y}^l, \boldsymbol{Y}^m]. \tag{23}$$

We call $\boldsymbol{A}$ the stability matrix for external modes. If the minimum eigenvalue of $\boldsymbol{A}$ is negative, then the plasma is unstable against ideal external kink modes. The matrix $\boldsymbol{A}$ also plays an important role in the stability of resistive wall modes [6].

The basis $\{\boldsymbol{Y}^m(r)\}$ can be constructed by using the response formalism [7]. Let us write $\boldsymbol{Y}^m(r)$ as

$$\boldsymbol{Y}^m(r) = \boldsymbol{X}^m(r) + \boldsymbol{Z}^m(r), \tag{24}$$

where $\boldsymbol{Z}^m(r)$ given analytically satisfies the inhomogenous boundary condition, eq. (19). Then, we have an inhomogeneous equation for $\boldsymbol{X}^m(r)$ with the homogenous boundaryn condition

$$\mathcal{N}\boldsymbol{X}^m(r) = -\mathcal{N}\boldsymbol{Z}^m(r), \quad \boldsymbol{X}^m(a) = 0. \tag{25}$$

Since eq. (25) can be solved by the MARG2D code, we can construct the basis $\{\boldsymbol{Y}^m(r)\}$ and the stability matrix $\boldsymbol{A}$.

The present formalism has been satisfactory applied to the stability analysis of low $n$, typically $n = 1$ or 2, external modes, which is reported at this conference [8].

## 4. Extension of the MARG2D formulation into the vacuum

The vacuum contribution to the change of potential energy is represented by the matrix $\boldsymbol{M}_V$ in eq. (14), which is computed by using Green's function of the Laplace operator [9]. This method deals flexibly with the shape of a conducting wall. However, the method is limited to low $n$ modes since it is difficult to evaluate numerically special functions that appear in constructing $\boldsymbol{M}_V$ when $n$ becomes large. It is convenient to express magnetic fields by a vector potential for middle $n$ ($n$ = 2, 3, . . . , 10) or high $n$ ($n$ > 10) external modes, which was shown in ref. [10]. We adopt the vector potential method in the MARG2D code for the analysis of peeling modes. It is shown that the Lagrangian density function for the change of energy in the vacuum has the same form as that for the change of plasma potential energy, and that the MARG2D formulation is easily extended to such external modes.

Let $\psi_V(R, Z)$ be a function defined in the vacuum. The coutour $\psi_V(R, Z) = \psi_s$ coincides with the plasma surface $\psi(R, Z) = \psi_s$. We also assume that the outermost contour of $\psi_V(R, Z)$ = const. coincides with the cross section of the conducting wall. Next, we introduce a vector field $\boldsymbol{C}_V$ in the vacuume by

$$\boldsymbol{C}_V = \nabla\phi \times \nabla\psi_V + T_V(R, Z)\nabla\phi. \quad (26)$$

Here we assume $T_V$ is independent of $\phi$. It is easy to see that $\boldsymbol{C}_V$ is a solenoidal vector [10], div $\boldsymbol{C}_V = 0$. The poloidal angle $\theta$ and the function $T_V$ can be defined such that

$$\frac{\boldsymbol{C}_V \cdot \nabla\phi}{\boldsymbol{C}_V \cdot \nabla\theta} = q_s (= \text{const.}), \quad (27)$$

at all points in the vacuum, where $q_s$ is the safety factor at the edge.

The perturbation of magnetic fields is given by

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \quad (28)$$

and the vector potential $\boldsymbol{A}$ is expressed by

$$\boldsymbol{A} = \vec{\xi}_V \times \boldsymbol{C}_V, \quad (29)$$

where $\vec{\xi}_V$ is the unknown vector to be determined. By introducing the functions

$$Y(\psi_V, \theta) := \vec{\xi}_V \cdot \nabla\psi_V, \quad (30)$$

$$V(\psi_V, \theta) := \vec{\xi}_V \cdot \nabla\theta - \frac{1}{q_s}\vec{\xi}_V \cdot \nabla\phi, \quad (31)$$

the change of energy in the vacuum is given by

$$W_V = \pi q_s \int L \mathrm{d}\psi_V \mathrm{d}\theta, \quad (32)$$

$$L = a\left|D_\theta(Y)\right|^2 + T\left|\frac{\partial V}{\partial\theta} + \frac{\partial Y}{\partial\psi_V}\right|^2 + b\left|inq_s V + \frac{\partial Y}{\partial\psi_V}\beta_{\psi\theta}D_\theta(Y)\right|^2. \quad (33)$$

Here

$$D_\theta(Y) := \left(\frac{1}{q_s}\partial_\theta - in\right)Y, \quad (34)$$

$$\beta_{\psi\theta} := q_s\frac{\nabla\psi_V \cdot \nabla\theta}{\left|\nabla\psi_V\right|^2}, \quad (35)$$

and

$$a := \frac{T_V}{R^2\left|\nabla\psi_V\right|^2}, \quad b := \frac{\left|\nabla\psi_V\right|^2}{T_V}. \quad (36)$$

We can eliminate the function $V$ by the same procedure in Sec. **2** by using the poloidal Fourier harmonics

$$Y(\psi_V, \theta) = \sum_l Y_l(\psi_V)\exp(\mathrm{i}l\theta), \quad (37)$$

$$\boldsymbol{Y}(\psi_V) := \{Y_{-L_f}(\psi_V), \ldots, Y_{L_f}(\psi_V)\}^t. \quad (38)$$

The reduced form has the same form in eq. (11):

$$W_V = 2\pi^2 q_s \int T_V(\psi_V\ L(\boldsymbol{Y}, \boldsymbol{Y})\mathrm{d}\psi_V, \quad (39)$$

$$L(\boldsymbol{Y}, \boldsymbol{Y}) = \left\langle\frac{\mathrm{d}\boldsymbol{Y}}{\mathrm{d}\psi_V}\middle|\boldsymbol{L}\middle|\frac{\mathrm{d}\boldsymbol{Y}}{\mathrm{d}\psi_V}\right\rangle + \left\langle\boldsymbol{Y}\middle|\boldsymbol{K}\middle|\boldsymbol{Y}\right\rangle + \left\langle\frac{\mathrm{d}\boldsymbol{Y}}{\mathrm{d}\psi_V}\middle|\boldsymbol{M}^t\middle|\boldsymbol{Y}\right\rangle + \left\langle\boldsymbol{Y}\middle|\boldsymbol{M}\middle|\frac{\mathrm{d}\boldsymbol{Y}}{\mathrm{d}\psi_V}\right\rangle. \quad (40)$$

## 5. Summary

We have reported the recent progress in the theory of the Newcomb equation and applications to MHD stability analysis. A theory of external modes has been developed by using the Newcomb equation, which is also useful for the analysis of resistive wall modes. The MARG2D formulation has been extended in the vacuum region. Such extension enable us to analyze the stability of the edge region of a tokamak, and code developing is now on going.

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