Relation between the Radial Electric Field and the Flow 
in a Tokamak Plasma with \( \nabla T = 0 \)

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Abstract

The toroidal viscosity determines the neoclassical radial electric field in an axisymmetric tokamak plasma as in non-axisymmetric systems. This toroidal viscosity is very small and obtained only when the finite orbit width effect is considered. In the case of uniform temperature (VT = 0), the wave equation is derived to show the oscillatory behaviour of the radial electric field. The oscillatory radial electric field converges a steady state which satisfies the satandard neoclassical relation between the parallel flow and the radial electric field.

Keywords:

neoclassical radial electric field, tokamak, non-axisymmetric system, finite orbit width, toroidal viscosity, parallel flow

1. Introduction

It is a very important subject to examine what mechanism determines the radial electric field in toroidal systems, because the radial electric field plays an important role for the plasma transport. The radial electric field is determined by the ambipolar condition: \( \Gamma_r (E_r) = \Gamma_i (E_i) \), where \( \Gamma_r \) and \( \Gamma_i \) are electron and ion particle fluxes, respectively, and \( E_r \) is the radial electric field. The particle fluxes arise from various processes such as neoclassical transport, anomalous transport, charge exchange, particle loss out the of confinement region.

In the present paper, we consider only neoclassical transport to determine the radial electric field in axisymmetric tokamaks. It is well known, in axisymmetric tokamaks, that the particle transport is intrinsically ambipolar, which means that the electron and ion particle fluxes are independent of the radial electric field, and thus it is not determined. However, this intrinsically ambipolar condition holds only in the lowest order of \( \varepsilon \) in the drift kinetic equation, where \( \varepsilon = p_L / L \ll 1 \) with \( p_L \) the poloidal Larmor radius and \( L \) the characteristic length of the plasma. The standard neoclassical theory \(^1,2\) holds in this lowest order of \( \varepsilon \) and thus in the limit of the small orbit width (SOW) of the plasma particles. Therefore, to determine the neoclassical radial electric field \( E_r \) in an axisymmetric tokamak, the fourth order \( O(\varepsilon^4) \) of particle flux must be calculated solving the drift kinetic equation up to the second order \( O(\varepsilon^2) \). Assuming the constant tempearature (VT = 0), Rosenbluth et al. \(^3\) calculated the fourth order particle flux and obtained the radial electric field at the steady state. This fourth order particle flux is very small and attributed to the finite orbit width (FOW) effect.

In the general case of non-zero VT, it is difficult to obtain higher order flux without solving numerically the drift kinetic equation. To include comprehensively the FOW effect in studying neoclassical transport, the \( \delta f \) Monte Carlo method has been developed \(^4-11\). Since this method includes accurately both FOW effect and collisional effect which ensures the momentum and energy conservations, it is applicable to the neoclassical transport independent of \( \varepsilon \) ordering. Recently, Okamoto et al. \(^11\) calculated the neoclassical radial electric field in a tokamak plasma with a flow by using a \( \delta f \) simulation code which solves simultaneously the drift kinetic equation and the equation for the time development of the radial electric field. The radial electric field oscillates with a frequency of the geodesic acoustic mode and converges to a steady state value.

In the present paper, we derive the wave equation of the oscillatory electric field in the case of VT = 0, assuming that the radial particle flux is calculated by the \( \delta f \) simulation. It is shown that the oscillatory electric field converges and the resultant steady state radial electric field and the parallel flow has the same relation with that of the standard neoclassical theory. It is conjectured that the standard neoclassical relationship between the radial electric field and the parallel flow holds even for the plasma with FOW.

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2. Parallel and toroidal viscosities

The equation for momentum balance is given, at the steady state, by

\[ n_e e_\alpha \left( \vec{E} + \vec{u}_\alpha \times \vec{B} \right) + \vec{F}_\alpha - \nabla P_e - \nabla \cdot \Pi = 0 \]  

(1)

Taking toroidal component \((R^2 \nabla \phi)\) gives the particle flux in the radial direction for species \(\alpha\),

\[ \Gamma_{\alpha \psi} = \left\langle n_{\alpha} \vec{u}_\alpha \cdot \nabla \psi \right\rangle = \left\langle R^2 \nabla \phi \cdot n_{\alpha} \vec{u}_\alpha \times \vec{B} \right\rangle \]  

(2)

\[ \Gamma_{\alpha \nu} = -\left\langle R^2 \nabla \phi \cdot \frac{1}{e_\alpha} \vec{F}_\alpha + n_{\alpha} \vec{E} \right\rangle \]  

\[ + \frac{1}{e_\alpha} \left\langle \nabla \cdot \left( R^2 \nabla \phi \cdot \Pi \right) \right\rangle \]  

(3)

It is noted that the relations \(\sum F_\alpha = 0\) (momentum conservation during the collision), \(\Sigma e_\alpha n_{\alpha} = 0\) (charge neutrality), and \(\Sigma \Pi_\alpha = 0\) (viscosity force balance) hold. Then, \(\sum e_\alpha \Gamma_{\alpha \psi} = 0\), which is consistent with vanishing radial current at the steady state.

In the axisymmetric tokamak, the toroidal viscosity is small due to axisymmetry, which is on the order of \(O(\epsilon^2)\) if \(\epsilon\) is small [3], and parallel force balance \(\dot{B}(\dot{B} \cdot \dot{F}_\phi) = (\dot{B} \cdot \nabla \cdot (R^2 \Phi))\) dominates the particle diffusion. Then, in the lowest order, the relation \(\Gamma_{\alpha \psi} = \Gamma_{\alpha \nu}\) holds independently of the radial electric field \(E_\psi\). This is the intrinsic ambipolarity.

The total particle flux in the radial direction is given, omitting the subscript \(\psi\), by

\[ \Gamma = -\Gamma_{\alpha e} - \Gamma_{\alpha i} + \Gamma_{\alpha \nu} + \Gamma_{\alpha \psi} \]  

(4)

Here, \(\Gamma_{\alpha e}, \Gamma_{\alpha i}, \Gamma_{\alpha \nu}, \) and \(\Gamma_{\alpha \psi}\) are radial particle fluxes caused by electron-electron, electron-ion, ion-ion, and ion-electron collisions, respectively. In the lowest order, \(\Gamma_{\alpha i} = \Gamma_{\alpha \psi}\) independent of \(E_\psi\), and \(\Gamma_{\alpha \nu}\) is very small compared to \(\Gamma_{\alpha \psi}\), since the particle fluxes due to self-collisions are attributed to FOW effects in the axisymmetric tokamak. \(\Gamma_{\alpha i}\) comes from the toroidal viscosity \(\Gamma_{\alpha i} = \left\langle \nabla \cdot (R^2 \nabla \Phi \cdot \Pi) \right\rangle\), which depends on \(E_\psi\). It is concluded that \(\Gamma_{\alpha i}(E_\psi) = 0\) determines \(E_\psi\).

In non-axisymmetric tori (stellarator, helical system like LHD), the toroidal viscosity \(\langle \dot{B}_r \cdot \nabla \cdot (R^2 \Phi) \rangle\) dominates the particle diffusion over the parallel viscosity and \(\Gamma_{\alpha \psi}(E_\psi) = \Gamma_{\alpha \psi}(E_\psi)\) determines \(E_\psi\).

3. Wave equation

Only ions are treated in the present paper. We consider a plasma in an axisymmetric tokamak and all the variables depend only on the radial position \(\psi\) and the poloidal angle \(\theta\). In the coordinates \((E, \mu, \psi, \theta)\), the drift kinetic equation is given by [1]

\[ \frac{df}{dt} = \frac{e}{m} \frac{d\Phi}{dt} \frac{\delta f}{\delta \mu} + \left( \vec{v}_i + \vec{v}_d \right) \cdot \nabla f = C(f, f) \]  

(5)

where \(\vec{v}_i\) is the parallel velocity, \(\vec{v}_d\) is the drift velocity, and \(C\) is the collision operator. This drift kinetic equation is solved numerically by the \(\delta f\) method with two weights [5]. However, to solve this equation, the equation for the radial electric field is required.

From Poisson equation and the continuity equation, the time development for the radial electric field is given by

\[ \left\langle \frac{|
abla \psi|^2}{B^2} \right\rangle + 4\pi n_e e^2 \left\langle \left( \frac{|
abla \psi|^2}{B^2} \right) \right\rangle \frac{\partial^2 \Phi}{\partial \psi^2} = 4\pi n_e \Gamma_i \]  

(6)

In this equation, \(\Gamma_i = \Gamma_{\alpha \psi}\) is the ion particle flux. \((\Gamma_{\alpha \psi}\) is very small and neglected.) \(\Phi\) is the electric potential, \(\psi\) is the toroidal flux (the radial electric field is \(E_\psi = -\partial \Phi / \partial \psi\)), and the second term on the left hand side is attributed to the classical polarization current. The bracket \(\langle \cdot \rangle\) means the flux average. The ion particle flux \(\Gamma_i\) is defined by

\[ \Gamma_i = \left\langle \int d\psi \left( \vec{v}_i \cdot \nabla \psi \right) f \right\rangle \]  

(7)

Equation (6) with (7) holds independently of the magnitude of pressure gradients. That is, eqs.(6) and (7) are unrelated to the ordering of \(\epsilon = \rho_p / L\). In the standard neoclassical theory, like-particle collisions produce no particle flux in the SOW limit, and hence \(\Gamma_i\) vanishes. However, in the \(\delta f\) simulation, \(\Gamma_i\) remains finite even if the pressure gradient is very small, because the effect of FOW is rigorously taken into account by solving guiding center equations. If the FOW effect is small, \(\Gamma_i\) is small and a small \(E_\psi\) is generated. If the FOW effect is large, \(\Gamma_i\) is large and a large \(E_\psi\) is generated. The radial electric field evolves so as to vanish \(\Gamma_i\) to maintain the charge neutrality.

In this paper, only the case of \(\nabla \psi \neq 0\) is considered. For simplicity, it is assumed that the magnetic surfaces are concentric and circular. Then, equation (7) becomes

\[ \left( 1 + \frac{\epsilon^2}{v_A^2} \right) \frac{\partial E_\psi}{\partial t} = -4\pi e \left\langle \int d^2 \psi v_d \right\rangle \]  

(8)

where \(v_A = B/(\sqrt{4\pi n e})\) is the Alfvén speed. The radial and poloidal drift velocities are given by

\[ v_d = -\frac{v_i^2 + v_\theta^2}{2\Omega_0 R_0} \sin \theta \]  

(9)

\[ v_d = -\frac{E_\psi}{B} \frac{v_i^2 + v_\theta^2}{2\Omega_0 R_0} \sin \theta \]  

(10)

where \(R_0\) is the major radius and \(\Omega_0 = eB_0/mc\) with \(B_0\) the magnetic field at the axis. The parallel flow velocity has a form of
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\[ v_1(r, \theta) = \left\langle v_1 \right\rangle \frac{R_0}{B} \]  

(11)

with

\[ B = B_0 \left( 1 - \frac{r}{R} \cos \theta \right). \]  

(12)

Taking the time derivative of Eq. (8) yields

\[ \epsilon \frac{\partial^2 E_r}{\partial t^2} = -4 \pi \epsilon \left\langle \int d^3 v \frac{\partial f}{\partial v} v_n \right\rangle \]  

(13)

with

\[ E_r \equiv 1 + \frac{e^2}{v_n^2} \simeq \frac{e^2}{v_n^2}. \]  

(14)

From the drift kinetic equation (5)

\[ \frac{\partial f}{\partial t} = -\frac{e}{m} \frac{\partial \Phi}{\partial t} \frac{\partial f}{\partial v} - \left( \dot{v}_n^2 + \vec{v}_n \right) \cdot \nabla f + C. \]  

(15)

The standard neoclassical theory gives an exact solution to the drift kinetic equation for the case of VT = 0 [1]. The solution is a shifted-Maxwellian distribution function \( f_{SM} \) expressed as,

\[ f_{SM} = \left( 1 + 2 \frac{v_n v_1}{v_a^2} \right) f_M \]  

(16)

where \( f_M \) is a Maxwellian with a thermal speed \( v_n \) given by

\[ f_M = \frac{n}{\pi v_n^3} \exp \left( \frac{e \Phi}{T} - \frac{2 \epsilon}{m v_n^2} \right) \]  

(17)

Here, \( T \) is the ion temperature. It is noted that the shifted Maxwellian annihilates off the collision term \( C(f_{SM}, f_{SM}) = 0 \).

Inserting Eq. (16) together with Eq. (17) into Eq. (13) yields, up to the order of \( O(\epsilon = r/R_0) \) and \( O(\dot{v}_n/v_n) \),

\[ \frac{\partial^2 E_r}{\partial t^2} + \omega_{GAM}^2 E_r = H \]  

(18)

where \( \omega_{GAM} \) is the GAM (Geodesic Acoustic Mode) frequency [12] given by

\[ \omega_{GAM}^2 = \frac{7}{4} \frac{v_n^2}{R_0^2} \]  

(19)

and

\[ H = \omega_{GAM}^2 \left( \frac{T}{e} \frac{1}{n} \frac{dn}{dr} + \frac{m \Omega}{e q R_0} \left\langle v_1 \right\rangle \right) \]  

(20)

where \( q \) is the safety factor, \( m \) the ion mass, \( e \) the ionic charge, and \( \Omega \) is the ion cyclotron frequency. The wave equation is the same as that in ref. [11] in the small \( \left\langle v_1 \right\rangle \) limit.

Equation (18) suggests that the radial electric field interacts with moving particles to generate an oscillation with a frequency \( \omega_{GAM} \). The solution to eq. (18) is

\[ E_r = E_0 + A e^{-i \omega_{GAM} t}. \]  

(21)

The amplitude A is Landau damped and \( E_0 \) is changed slowly by magnetic pumping [13]. It is known that the damping rate depends strongly on the value of the safety factor \( q \). In the long time limit after the GAM oscillation damps out or after time averaging, the relation between the radial electric field and the flux averaged parallel flow becomes

\[ \left\langle v_1 \right\rangle = \frac{q R_0 v_n^2}{2 r \Omega} \left( \frac{1}{n} \frac{dn}{dr} - \frac{e}{T} E_r \right). \]  

(22)

This is just the relation given by the standard neoclassical theory [1]. Note that the equation (18) or (21) has been derived without SOW assumption.

4. Summary

The analysis has been restricted to the plasma with no temperature gradients (VT = 0). The purpose of this restriction is to compare the relation of the radial electric field and the parallel flow between the standard neoclassical theory in the SOW limit and the present analysis assuming no SOW limit. Both relations agree completely. This suggests that the FOW effect may not affect the standard relation between the radial electric field and the parallel flow velocity.

References

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