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Quantum Chaos Theory and the Spectrum of Ideal-MHD Instabilities in Toroidal Plasmas

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Abstract

In a fully 3-D system such as a stellarator, the toroidal mode number n ceases to be a good quantum number – all ns within a given mode family being coupled. It is found that the discrete spectrum of unstable ideal MHD (magnetohydrodynamic) instabilities ceases to exist unless MHD is modified (regularized) by introducing a short-perpendicular-wavelength cutoff. Attempts to use ray tracing to estimate the regularized MHD spectrum fail due to the occurrence of chaotic ray trajectories. In quantum chaos theory, strong chaos in the semiclassical limit leads to eigenvalue statistics the same as those of a suitable ensemble of random matrices. For instance, the probability distribution function for the separation between neighboring eigenvalues is as derived from random matrix theory and goes to zero at zero separation. This contrasts with the Poissonian distribution found in separable systems, showing that a signature of quantum chaos is level repulsion. In order to determine whether eigenvalues of the regularized MHD problem obey the same statistics as those of the Schrödinger equation in both the separable 1-D case and the chaotic 3-D cases, we have assembled data sets of ideal MHD eigenvalues for a Suydam-unstable cylindrical (1-D) equilibrium using *Mathematica* and a Mercier-unstable (3-D) equilibrium using the CAS3D code. In the 1-D case, we find that the unregularized Suydam-approximation spectrum has an anomalous peak at zero eigenvalue separation. On the other hand, regularization by restricting the domain of $\boldsymbol{k}_\perp$ recovers the expected Poissonian distribution. In the 3-D case we find strong evidence of level repulsion within mode families, but mixing mode families produces Poissonian statistics.

Keywords:

quantum chaos, ideal MHD, interchange spectrum, Suydam, finite Larmor radius, eigenvalue spacing, probability distribution

1. Introduction

In ideal MHD the spectrum of the growth rates, γ, of instabilities is difficult to characterize mathematically because the linearized force operator is not compact [1]. This gives rise to the possibility of a dense set of accumulation points (descriptively called the "accumulation continuum" by Spies and Tataronis [2] though more correctly termed [3] the *essential spectrum*).

The continuous spectrum in quantum mechanics arises from the unboundedness of configuration space, whereas the MHD essential spectrum arises from the unboundedness of Fourier space – there is no minimum wavelength in ideal MHD. This is an unphysical artifact of the ideal MHD model because, in reality, low-frequency instabilities with $|\boldsymbol{k}_\perp|$ much greater than the inverse of the ion Larmor radius, a_i, cannot exist (where $\boldsymbol{k}_\perp$ is the projection of the local wavevector into the plane perpendicular to the magnetic field $\boldsymbol{B}$).

Perhaps the greatest virtue of ideal MHD in fusion plasma physics is its mathematical tractability as a first-cut model for assessing the stability of proposed fusion-relevant experiments with complicated geometries. For this purpose a substantial investment in effort has been expended on developing numerical matrix eigenvalue programs, such as the three-dimensional (3-D) TERPSICHORE [4] and CAS3D [5] codes. These solve the MHD wave equations for perturbations about static equilibria, so that the eigenvalue $\omega^2 \equiv -\gamma^2$ is real due to the Hermiticity (self-adjointness [6]) of the linearized force and kinetic energy operators. They use finite-element or finite-difference methods to convert the infinite-dimensional Hilbert-space eigenvalue problem to an approximating finite-dimensional matrix problem.

In order properly to verify the convergence of these codes in 3-D geometry it is essential to understand the nature

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of the spectrum – if it is quantum-chaotic then convergence of individual eigenvalues cannot be expected and a statistical description must be used.

It is the thesis of this paper that the language of quantum chaos [7] theory indeed provides such a statistical framework for characterizing MHD spectra in that it seeks to classify spectra statistically by determining whether, and to what degree, they belong to various universality classes.

In the cylindrical case the eigenvalue problem is separable into three one-dimensional (1-D) eigenvalue problems, with radial, poloidal, and toroidal (axial) quantum numbers l, m, and n, respectively. It is thus to be expected *a priori* that the spectrum will fall within the standard quantum chaos theory universality class for integrable, non-chaotic systems [7]. In particular, it is to be expected that the probability distribution function for the separation of neighboring eigenvalues is a Poisson distribution. However, the nature of the MHD spectrum is quite different from that of the typical quantum, microwave and acoustic systems normally dealt with in quantum chaos theory and it is necessary to test this conjecture by explicit calculation. In fact we find that the result depends on the method of regularization.

We first present the eigenvalue equation for a reduced MHD model of a large-aspect-ratio (effectively cylindrical) stellarator. We study a plasma in which the Suydam criterion [8] for the stability of interchange modes is violated, so the number of unstable modes tends to infinity as the small-wavelength cutoff tends to zero. To compute large-m eigenvalues we transform to a Schrödinger-like form of the radial eigenvalue equation [9], which has essentially the same form in configuration (r) space as in Fourier (k_r) space, thus allowing easy regularization by restricting the k_r domain. To simplify even further we approximate the effective potential by a parabola, thus yielding the quantum harmonic oscillator equation, solvable in parabolic cylinder functions [10].

Real, finite-aspect-ratio stellarators are fully 3-D and their ideal-MHD spectra may be expected *a priori* to fall within the universality class appropriate to time-reversible quantum chaotic systems, where the spectral statistics are found to be the same as for a Gaussian orthogonal ensemble of random matrices [7] in regions where ray tracing reveals chaotic dynamics [11]. At the end of this paper we give a brief report of 3-D calculations peformed with the CAS3D code on a Mercier-unstable, high-mirror-ratio, high-iota equilibrium representing a Wendelstein 7-X (W7-X) stellarator variant [12].

2. One-dimensional model eigenvalue equation

In this paper we study an effectively circular-cylindrical MHD equilibrium, using cylindrical coordinates such that the magnetic axis coincides with the z-axis, made topologically toroidal by periodic boundary conditions. Thus z and the toroidal angle ζ are related through $\zeta \equiv z/R_0$, where R_0 is the major radius of the toroidal plasma being modeled by this cylinder. The poloidal angle θ is the usual geometric cylindrical angle and the distance r from the magnetic axis labels the magnetic surfaces (the equilibrium field being trivially integrable in this case). The plasma edge is at $r = a$.

In the cylinder there are two ignorable coordinates, θ and ζ, so the components of $\boldsymbol{\xi}$ are completely factorizable into products of functions of the independent variables separately. In particular, we write the r-component as

$$r\xi_r = \exp(\mathrm{i}m\theta)\exp(-\mathrm{i}n\zeta)\varphi(r)\,, \qquad (1)$$

where the periodic boundary conditions quantize m and n to integers and we choose to work with the stream function $\varphi(r) \equiv r\xi_r(r)$.

Since the primary motivation of this paper is stellarator physics, we use the reduced MHD ordering for large-aspect stellarators [13,14], averaging over helical ripple to reduce to an equivalent cylindrical problem [15,16]. The universality class should be insensitive to the precise choice of model as long as it exhibits the behavior typical of MHD instabilities in a cylindrical plasma, specifically the existence of interchange instabilities and the occurrence of accumulation points at finite growth rates.

Defining $\lambda \equiv \omega^2$ we seek the spectrum of λ-values satisfying the scalar equation

$$L\varphi = \lambda M\varphi \qquad (2)$$

under the boundary conditions $\varphi(0) = 0$ at the magnetic axis and $\varphi(1) = 0$, appropriate to a perfectly conducting wall at the plasma edge (using units such that $r = 1$ there).

The operator $M = -\nabla_\perp^2$ and L is given by

$$L = -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(n - m\iota)^2 r\frac{\mathrm{d}}{\mathrm{d}r} + \frac{m^2}{r^2}\left[(n - m\iota)^2 - \dot{\iota}^2 G + \frac{\ddot{\iota}}{m}(n - m\iota)\right], \qquad (3)$$

where G is a Suydam stability parameter (> 1/4 for instability [8]), proportional to the pressure gradient $p'(r)$ and the average field line curvature [14].

In this paper we use the notation $\dot{f} \equiv rf'(r)$ for an arbitrary function f, so $\dot{\iota} \equiv r\mathrm{d}\iota/\mathrm{d}r$ is a measure of the magnetic shear and $\ddot{\iota}$ measures the variation of the shear with radius.

We observe some differences between eq. (2) and the standard quantum mechanical eigenvalue problem $H\psi = E\psi$. One is of course the physical interpretation of the eigenvalue – in quantum mechanics the eigenvalue $E \equiv \hbar\omega$ is linear in the frequency because the Schrödinger equation is first order in time, whereas our eigenvalue λ is quadratic in the frequency because it derives from a classical equation of motion.

Another difference is that eq. (2) is a *generalized* eigenvalue equation because M is not the identity operator. This is one reason why it is necessary to treat the MHD spectrum explicitly rather than simply assume it is in the same universality class as standard quantum mechanical systems.

Equation (2) is very similar to the normal mode equation analyzed in the early work on the interchange growth rate in stellarators by Kulsrud [15]. However, unlike this and most other MHD studies we are concerned not with finding the highest growth rate, but in characterizing the complete set of unstable eigenvalues.

Suydam instabilities occur only for values of m and n such that $n - m\iota$ vanishes. For the 1-D numerical work in this paper we use a parabolic transform profile $\iota = \iota_0 + \iota_2 r^2$ as illustrated in Fig. 1.

Given a rational fraction $\mu = n_\mu/m_\mu$ in the interval $[\iota(0), \iota(a)]$ (where n_μ and m_μ are mutually prime) there is a unique radius r_μ such that $\iota(r_\mu) = \mu$. Any pair of integers $(m, n)_{\mu,\nu} \equiv (\nu m_\mu, \nu n_\mu)$, $\nu = 1, 2, 3, \ldots$ satisfies the resonance condition

$$n_{\mu,\nu} - m_{\mu,\nu}\iota(r_\mu) = 0\,. \qquad (4)$$

We use a broad pressure profile that is sufficiently flat near the magnetic axis that the Suydam instability parameter G goes to zero at the magnetic axis, and for which p' vanishes at the plasma edge. The resulting G-profile is shown in Fig. 2.

Defining a scaled radial variable $x = m(r - r_\mu)/r_\mu$, we can find the large-m spectrum of eq. (2) by expanding all quantities in inverse powers of m, and equating the LHS to zero order by order.

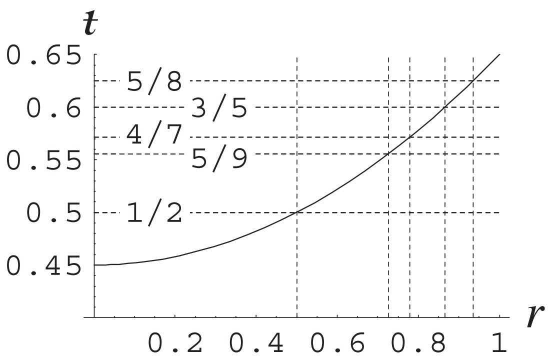

Fig. 1 The rotational transform $\iota(r) \equiv 1/q(r)$ with $\iota_0 = 0.45$, $\iota_2 = 0.2$. All distinct rational magnetic surfaces $\mu = n/m$ are shown for m up to 10.

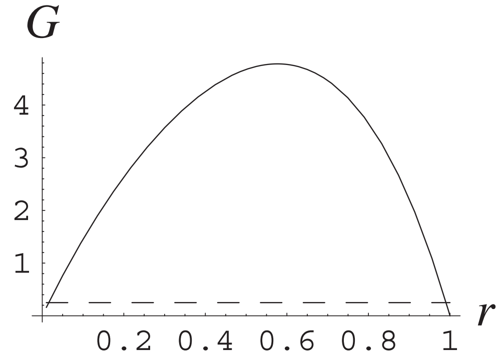

Fig. 2 The Suydam criterion parameter $G(r)$ (solid line), and the instability threshold 1/4 (dashed line), showing nearly all the plasma is Suydam unstable.

In this paper we work only to lowest order in $1/m$, the *Suydam approximation*. As found by Kulsrud [15], we have the generalized eigenvalue equation

$$\mathrm{L}^{(0)}\varphi^{(0)} \equiv \frac{r_\mu^2}{m^2}\left(L^{(0)} - \lambda^{(0)}M^{(0)}\right)\varphi^{(0)} = 0\,, \qquad (5)$$

where, more explicitly,

$$\frac{\mathrm{L}^{(0)}}{\iota'^2} = -\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + \Gamma^2)\frac{\mathrm{d}}{\mathrm{d}x} + x^2 + \Gamma^2 - G\,, \qquad (6)$$

with $\Gamma^2 \equiv -\lambda^{(0)}/\iota'^2$ and ι' and G evaluated at r_μ. Under the boundary conditions as $\varphi^{(0)} \to 0$ as $\gamma \to \pm\infty$, eq. (5) can be solved to give a square-integrable eigenfunction, with growth rate $\gamma = \iota'\Gamma$, provided $\lambda^{(0)} < 0$ is one of the eigenvalues $\lambda_{\mu,l}$. The radial mode number $l = 0, 1, 2, \ldots$ denotes the number of nodes of the eigenfunction $\varphi^{(0)} \equiv \varphi_{\mu,l}(r)$. Note that $\lambda_{\mu,l}$ depends only on $\mu = n/m$ and is otherwise independent of the magnitude of m and n.

Restricting attention to unstable modes, so that $\gamma \equiv (-\lambda)^{1/2}$ is real, we transform eq. (5) to the Schrödinger form [9]

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\eta^2} + Q(\eta)\psi = 0\,, \qquad (7)$$

where

$$Q \equiv G - \tfrac{1}{4} - \tfrac{1}{4}\mathrm{sech}^2\eta - \Gamma^2\cosh^2\eta\,, \qquad (8)$$

with η defined through $x \equiv \gamma\sinh\eta/\iota'(r_\mu)$, and $\psi \equiv (\cosh\eta)^{1/2}\varphi(x)$.

From, e.g., eq. (4.7) of [9] we see that, provided the Suydam criterion $G > 1/4$ is satisfied, there is an infinity of γ eigenvalues accumulating exponentially toward the origin from above (so the λ-values accumulate from below) in the limit $l \to \infty$.

Perhaps less widely appreciated (because m and n are normally taken to be fixed) is the fact that there is also a point of accumulation of the eigenvalues of eq. (2) at each $\lambda_{\mu,l}$ as $m \to \infty$ with l fixed. (Although $\lambda^{(0)}$ is infinitely degenerate, we can break this degeneracy by proceeding further with the expansion in $1/m$, thus showing that $\lambda_{\mu,l}$ is an accumulation point.) Since the rationals μ are dense on the real line, there is an "accumulation continuum" [2] between $\gamma = 0$ and the maximum growth rate, $\gamma = \gamma_{max}$.

3. Regularization

The accumulation points of the ideal MHD spectrum found above are mathematically interesting but exist only as a singular limit of equations containing more physics, including finite-Larmor-radius (FLR) effects and dissipation, that regularize the spectrum.

In order to proceed further we need to be explicit about the nature of this singular limit. As we are primarily concerned with the universality class question, we seek only a *minimal* modification of eq. (2) that has some physical basis but makes as little change to ideal MHD as possible. To preserve the Hermitian nature of ideal MHD we cannot use the drift correction used for estimating FLR stabilization of interchange modes by Kulsrud [15]. However it is possible to effect a pseudo-FLR regularization of ideal MHD by restricting $\boldsymbol{k}_\perp$ to a disk of radius less than the inverse ion Larmor radius. In our nondimensionalized, large-aspect ratio model this implies

$$(k_\theta^2 + k_r^2)^{1/2} \rho_* \le 1 , \qquad (9)$$

where k_r and k_θ are the radial and poloidal components of the wavevector, respectively, ρ_* and is the ion Larmor radius (at a typical energy) in units of the minor radius.

To apply eq. (9) precisely we need to relate k_r and k_θ to the eigenvalue problem discussed above. From eq. (1) we see that $k_\theta = m/r$. We define k_r as the Fourier variable conjugate to r. Fourier transformation of eq. (2) is only practical in the large-m limit, when modes are localized near the resonant surfaces $r = r_\mu$, which is why we have restricted the discussion to leading order in the $1/m$ expansion.

Using the stretched radial coordinate $x \equiv m(r - r_\mu)/r_\mu$ we define $k_r \equiv m\kappa/r_\mu$, where κ is the Fourier-space independent variable conjugate to x. With the substitutions $\mathrm{d}/\mathrm{d}x \mapsto \mathrm{i}\kappa$, $x \mapsto \mathrm{i}\,\mathrm{d}/\mathrm{d}\kappa$, and using the fact that $\kappa\mathrm{d}/\mathrm{d}\kappa$ and $(\mathrm{d}/\mathrm{d}\kappa)\kappa \equiv 1 + \kappa\mathrm{d}/\mathrm{d}\kappa$ commute, eq. (5) transforms to

$$\left[-\frac{\mathrm{d}}{\mathrm{d}\kappa}(1+\kappa^2)\frac{\mathrm{d}}{\mathrm{d}\kappa} + \Gamma^2(1+\kappa^2) - G\right]\varphi_\kappa = 0 . \qquad (10)$$

The transformation $\kappa = \sinh\eta$ then leads back to Eq. (7), with λ now to be interpreted as a distorted Fourier-space independent variable, rather than as a real-space coordinate !

Equation (9) implies that Eq. (10) is to be solved on the domain $-\kappa_{max} \le \kappa \le \kappa_{max}$ where

$$\kappa_{max}(\mu) \equiv \left[\left(\frac{r_\mu}{m\rho_*}\right)^2 - 1\right]^{1/2} . \qquad (11)$$

This exists provided $|m| < m_{max}$, where

$$m_{max}(\mu) \equiv r_\mu / \rho_* . \qquad (12)$$

Analogously to quantum mechanical box-quantization we use Dirichlet boundary conditions at $\pm\kappa_{max}$.

4. Spectral statistics in the 1-D case

As only the qualitative nature of the spectrum is important, we approximate the function Q by $Q(0) + \frac{1}{2}Q''(0)\eta^2$, so eq. (7) can be solved in parabolic cylinder functions [10]. We find the dispersion relation

$$\nu + \tfrac{1}{2} = \frac{G - \Gamma^2 - \frac{1}{2}}{(4\Gamma^2 - 1)^{1/2}} , \qquad (13)$$

where $\nu = l$ in the unregularized case, $\kappa_{max} = \eta_{max} = \infty$. In the even-$l$, regularized case ν may be found by solving for a zero of $M(\nu/2,\ 1/2,\ (4\Gamma^2 - 1)^{1/2}\eta_{max}^2/2)$, where M is Kummer's function. For $l = 0$, ν becomes exponentially small as $\eta_{max} \to \infty$, which allows an approximate regularization formula to be derived.

We study the spectrum between the maximum $l = 1$ growth rate, $\gamma_{max}(l = 1) \equiv \max_\mu \gamma_{\mu,1}$, and the maximum overall growth rate, $\gamma_{max} = \max_\mu \gamma_{\mu,0}$. Only the $l = 0$ modes exist in this range of γ, which corresponds to the range in μ between $\mu_{min} \approx 0.522$ and $\mu_{max} \approx 0.628$. Throughout this range Γ is > 1/2, so that Q has a single minimum [9] and the quadratic approximation of this section is appropriate. In this range there are only four low-order rationals n/m with $m < 10$.

Taking $\rho_* = 0.001$, all pairs of integer values m, n in the fan-shaped region $1 \le m \le m_{max}(n/m)$, $\mu_{min} \le n/m \le \mu_{max}$ were evaluated, giving an initial dataset of over 32,000 points (m, n). The corresponding set of unregularized eigenvalues was calculated by solving eq. (13) with $\nu = 0$ and the eigenvalues were sorted and numbered from the top to give the integrated density of states "staircase" function $N(\gamma)$.

The curve $0.3523 - 9.5733 \times 10^{-11} N^2 - 1.1625 \times 10^{-20} N^4$ was found to give a good fit to the smoothed behavior of this function. Inverting this function gives the smoothed function $\tilde{N}(\gamma)$ which is used to "unfold" [7] spectra by defining a new "energy eigenvalue" $E \equiv \tilde{N}(\gamma)$, such that $N(E)$ increases linearly on average.

This means that the average separation of eigenvalues is now unity, making comparison with spectra from other physical systems meaningful and allowing universal behavior to become apparent if present. However, Fig. 3 shows that the probability distribution of eigenvalue spacings s is far from universal for the unregularized Suydam spectrum, exhibiting a delta-function-like spike at $s = 0$. This is presumably because, although we have truncated the spectrum in m, we have not removed the degeneracies arising for low-order rationals μ in the range $\mu_{min} < \mu < \mu_{max}$.

Fig. 4 on the other hand shows that when a similar procedure is applied to the regularized spectrum (retaining only regularized eigenvalues above $\gamma_{max}(l = 1)$, the universal

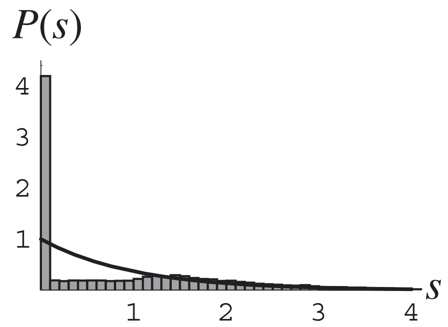

Fig. 3 The histogram shows an estimate, based on a data set of about 32,000 unregularized eigenvalues, of the probability distribution function for the eigenvalue separation s. The plot is dominated by the spike at $s = 0$.

Poisson distribution expected from a separable system is obtained to a good approximation, thus leading to the expectation that generic quantum chaos theory is applicable once any physically reasonable regularization is performed.

Further support for this hypothesis is obtained from a CAS3D study of a W7-X variant equilibrium with a non-monotonic, low-shear transform profile (ι_{axis} = 1.1066, ι_{min} = 1.0491, ι_{edge} = 1.0754). As seen from Fig. 5, when the statistics are analyzed within the three mode families the eigenvalue spacing distribution function is closer to the Wigner conjecture form found for generic chaotic systems [7] than to the Poisson distribution for separable systems, as might be expected from [11]. However, when the spectra from the three uncoupled mode familes are combined, there are enough accidental degeneracies that the spacing distribution becomes close to Poissonian.

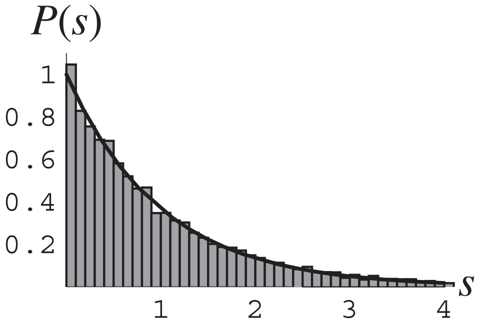

Fig. 4 Eigenvalue spacing distribution for a data set of about 12,000 regularized eigenvalues. The exponential curve shows the Poisson distribution.

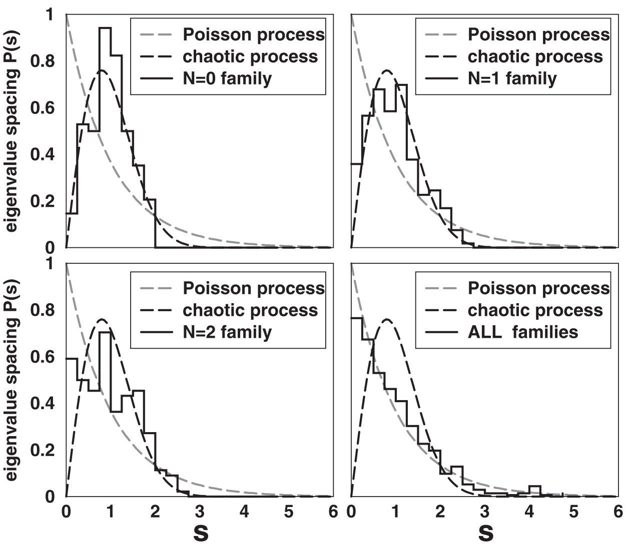

Fig. 5 Eigenvalue spacing distributions from mode family datasets N = 0 (137 values), N = 1 (214 values) and N = 2 (178 values) from a W7-X-like equilibrium, and the distribution for the combined spectrum, N = 0, 1 and 2.

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