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Effect of Finite Deviation of Super-Banana Orbit from Magnetic Surface on Neoclassical Transport in Helical Torus

TODOROKI Jiro, SANUKI Heiji and ITOH Kimitaka
National Institute for Fusion Science, Toki 509-5292, Japan

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Abstract

Neoclassical transport theory is extended to include the effect of finite deviation of the particle orbit from magnetic surfaces. In the low collisionality limit, the transport flux is expressed in the form of the integral with respect to the magnetic surfaces.

Keywords:
neoclassical transport, helical torus, trapped particle, adiabatic invariant, boundary layer

1. Introduction

The theory on transport process of the plasma in the toroidal configuration is based on the assumption that the fundamental plasma properties are described by several averaged quantities such as number density, temperature, and so on, which are considered to be functions of magnetic surfaces. This assumption implies that the plasma particles stay on the magnetic surface without collision, or more generally, the deviation of the orbit from the magnetic surface is much smaller than the characteristic length of the variation. In some experimental situations, however, the orbit of some particles deviates significantly from magnetic surfaces; the inclusion of such effect into the transport theory remains as an important issue of the theory.

In the collisionless limit, the distribution function is constant along the particle orbit in the phase space. Therefore the distribution function may be considered function of constant of motion, i.e. energy, magnetic moment, and the longitudinal adiabatic invariant. However, the statement that in the limit of low collisionality the distribution function is described only by the constant of motion is not correct in the toroidal plasma.

In the toroidal configuration there are several kinds of particles with orbit having the different topology. Since the collisional effects are described by the second order differential operator in the velocity space, continuity of the distribution function and its derivative with respect to the energy and magnetic moment is the requisite. The both requirements cannot be satisfied at the boundary of the different orbit topology. This means that, even if the collision frequency is small, the distribution function is not constant along the particle orbit in the certain part of the phase space, where the collisionality is effectively enhanced.

The purpose of this paper is to open the way to treat the effects of the finite deviation of the drift orbit from the magnetic surface. For that purpose, we shall first consider the simplest situation. The ripple trapped particle moves apart from the magnetic surface, while for the passing particles the orbit effects are ignored. The helical ripple is assumed small, and the number of trapped particles is small; therefore the lowest order distribution function is essentially determined by the majority passing particles. We also assume the up-down symmetry of the averaged quantities for the sake of the closing of the particle orbit.

Corresponding author's e-mail: todoroki@nifs.ac.jp

2. Bounce Averaged Kinetic Equation

We employ the Boozer's coordinates (ψ, θ, ϕ), ψ being the toroidal flux [1]. The magnetic field is expressed as

$$\boldsymbol{B} = \nabla\psi \times \nabla\theta + \nabla\phi \times \nabla\Psi_p(\psi)$$
$$= B_\psi \nabla\psi + B_\theta(\psi)\nabla\theta + B_\phi(\psi)\nabla\phi. \qquad (1)$$

We consider the helical torus with toroidal period N and the rotational transform per period is assumed to be small:

$$\iota/N \ll 1. \qquad (2)$$

The motion of charged particles in such systems are described by using the longitudinal adiabatic invariant [2]

$$J(E,\mu,\psi,\theta) = \frac{1}{2\pi}\oint P_\phi \,\mathrm{d}\phi, \qquad (3)$$

where

$$P_\phi \equiv m v_{/\!/}\frac{B_\phi}{B} + e\Psi_p(\psi). \qquad (4)$$

The representation of J depends on the nature of the periodic motion. The particles with the pitch angle

$$\lambda \equiv \frac{\mu}{E} \geq \lambda_t \equiv \frac{1}{B_{\max}(\psi,\theta)}, \qquad (5)$$

where $B_{\max}$ is the maximum of the magnetic field strength in the toroidal direction, are called as ripple trapped particles; they oscillate back and forth in toroidal direction within the helical ripple. The adiabatic invariant for them is

$$J = \frac{m}{2\pi}\oint \frac{v_{/\!/}B_\phi}{B}\,\mathrm{d}\phi \equiv \sqrt{E}\,J_r(\lambda,\psi,\theta). \qquad (6)$$

The particle having the pitch angle in the range $0 \leq 1 < \lambda_t$ are the passing particles, and they move in one direction, + or –, around the torus. The adiabatic invariant for them has the expression

$$J(E,\mu,\psi,\theta) = \frac{m}{2\pi}\int_0^{2\pi/N} \frac{v_{/\!/}B_\phi}{B}\,\mathrm{d}\phi \pm \frac{e}{N}\Psi_p$$
$$\equiv \sqrt{E}\,J_1(\lambda,\psi,\theta) \pm \frac{e}{N}\int_0^\psi \iota(\psi)\mathrm{d}\psi. \qquad (7)$$

Hence we can write the averaged kinetic equation as

$$\frac{1}{e}\frac{\partial J}{\partial\psi}\frac{\partial f}{\partial\theta} - \frac{1}{e}\frac{\partial J}{\partial\theta}\frac{\partial f}{\partial\psi} = \overline{C}(f,f). \qquad (8)$$

Here

$$\overline{C}(f,f) = \oint C(f,f)\frac{B}{v_{/\!/}}\,\mathrm{d}\phi \qquad (9)$$

is the bounce averaged collision operator.

3. Passing Particles

The passing particles are divided into two groups, according to the orbit, the circulating particles and the toroidally trapped particles. The circulating particles keep the same direction of motion along the magnetic field line, and stay in the same passing state while drifting transverse to the magnetic lines of force; the toroidally trapped particles may reach the point (transition point) where the parallel velocity along the magnetic field vanishes, drifting in poloidal direction. At the transition point, the toroidally trapped particle may change the direction of the motion along the magnetic field, or may transit to the ripple trapped state.

As we consider the small Larmor radius case, the first term in RHS of eq. (7) may be ignored in the lowest order approximation. Hence we can write the kinetic equation for the passing particles as.

$$\frac{\iota}{N}\frac{\partial f_0}{\partial\theta} = \overline{C}(f_0, f_0) = 0. \qquad (10)$$

The distribution function can be written as

$$f = f_0(w,\psi)\{1 + h(\lambda,\theta;w,\psi)\}, \quad |h| \ll 1 \qquad (11)$$

where f_0 is the local Maxwellian with the number density $n(\psi)$ and temperature $T(\psi)$, and w stands for the kinetic energy. If we only retain the pitch angle scattering term in the collision integral, the linearized collision term can be written as

$$\overline{C}(f_0 h, f_0) = \nu_{coll}\frac{\partial}{\partial\lambda}\lambda M\frac{\partial h}{\partial\lambda}f_0,$$
$$M \equiv \oint \frac{m v_{/\!/}}{B}B_\phi\,\mathrm{d}\phi. \qquad (12)$$

The linearized equation for passing particles is quite similar to the axisymmetric tokamak; the main body of the circulating particles and toroidally trapped particles are considered as collisionless, and the boundary layer between both types of particles are treated as is described in ref. [3].

Attached to the boundary with the ripple-trapped particles, a boundary layer develops. The analysis shows that the width of the layer is proportional to the collision frequency, and derives the consistent boundary conditions for the ripple-trapped particles at the boundary $\lambda = \lambda_t(\theta)$:

$$f^T = f_0\left\{1 + h\big|_{\lambda\to\lambda_t}\right\}, \ \frac{\partial f^T}{\partial \lambda} = -\frac{N}{\iota e}\frac{\left.\frac{\partial J}{\partial \theta}\right|_{\lambda\to\lambda_t}}{\frac{\partial \lambda_t}{\partial \theta}}\frac{\partial f_0}{\partial \psi}\,. \quad (13)$$

4. Ripple Trapped Particles

The characteristic time for the ripple-trapped particles is longer than that of passing particles. When the excursion time of the trapped particles ω_T^{-1} is much longer than the effective collision time ν_{eff}^{-1}, the effect of the finite orbit deviation is not essential, and the flux proportional to ν_{eff}^{-1} is found. The opposite limit is the most interesting case, and it is the main object of this paper.

The main body of the ripple-trapped particles is considered as collisionless, and the distribution function f^T is expanded as

$$f^T = F^{(0)} + (\nu/\omega_T)F^{(1)} + \cdots \quad (14)$$

where $F^{(0)} = F^{(0)}(J,\lambda)$. The first order function $F^{(1)}$ satisfies

$$\frac{1}{e}\frac{\partial J}{\partial \theta}\frac{\partial F^{(1)}}{\psi} - \frac{1}{e}\frac{\partial J}{\partial \psi}\frac{\partial F^{(1)}}{\partial \theta} = \nu\frac{\partial}{\partial \lambda}\lambda M\frac{\partial F^{(0)}}{\partial \lambda}\,. \quad (15)$$

If we introduce the action variable $H = H(J,\lambda)$ and angle variable ϑ, this equation can be rewritten in the form

$$K(J,\lambda)\frac{\partial F^{(1)}}{\partial \vartheta} = \mathscr{L}F^{(0)} \equiv \nu\frac{\partial}{\partial \lambda}\lambda M\frac{\partial F^{(0)}}{\partial \lambda}, \quad (16)$$

where

$$K(J,\lambda) \equiv \frac{1}{e}\frac{\partial J}{\partial \psi}\frac{\partial \vartheta}{\partial \theta} - \frac{1}{e}\frac{\partial J}{\partial \theta}\frac{\partial \vartheta}{\partial \psi}\,. \quad (17)$$

Carrying out the differentiation with respect to λ, we can write the RHS of eq. (16) in the form

$$\mathscr{L}F^{(0)} \equiv a_2\frac{\partial}{\partial \lambda}\lambda\frac{\partial F^{(0)}}{\partial \lambda} + a_1\lambda\frac{\partial F^{(0)}}{\partial \lambda} + b_3\frac{\partial^2 F^{(0)}}{\partial H\partial \lambda} + b_2\frac{\partial^2 F^{(0)}}{\partial H^2} + b_1\frac{\partial F^{(0)}}{\partial H}\,. \quad (18)$$

The coefficients a_1, a_2, b_1, b_2 and b_3 are functions of H, λ, and ϑ. The solvable condition of eq. (15) yields

$$\langle\mathscr{L}\rangle_\vartheta F^{(0)} \equiv \langle a_2\rangle_\vartheta\frac{\partial}{\partial \lambda}\lambda\frac{\partial F^{(0)}}{\partial \lambda} + \langle a_1\rangle_\vartheta\lambda\frac{\partial F^{(0)}}{\partial \lambda} + \langle b_2\rangle_\vartheta\frac{\partial^2 F^{(0)}}{\partial H^2} + \langle b_1\rangle_\vartheta\frac{\partial F^{(0)}}{\partial H}\langle b_3\rangle_\vartheta\frac{\partial^2 F^{(0)}}{\partial H\partial \lambda} = \frac{1}{e}K(J,\lambda)\Delta F \quad (19)$$

where

$$\langle A\rangle_\vartheta \equiv \frac{1}{\vartheta_2 - \vartheta_1}\int_{\vartheta_2(H,\lambda)}^{\vartheta_2(H,\lambda)} A\,\mathrm{d}\vartheta, \quad (20)$$

and ΔF denotes the difference of distribution function at two boundary points.

Now we shall try to solve eq. (19) in the expansion

$$F^{(0)}(H,\lambda) = F_0^{(0)}(H) + F_1^{(0)}(H,\lambda) + \cdots \quad (21)$$

with respect to some small parameter. This means that we have the following equations

$$\mathscr{L}^{(0)}F_0^{(0)} = 0\,, \quad (22)$$

$$\mathscr{L}^{(0)}F_1^{(0)} = -\langle b_2\rangle_\vartheta\frac{\partial^2 f_0}{\partial H^2} - \langle b_1\rangle_\vartheta\frac{\partial f_0}{\partial H} - \frac{1}{e}K(J,\lambda)\Delta F, \quad (23)$$

where

$$\mathscr{L}^{(0)} \equiv \langle a_2\rangle_\vartheta\frac{\partial}{\partial \lambda}\lambda\frac{\partial}{\partial \lambda} + \langle a_1\rangle_\vartheta\frac{\partial}{\partial \lambda}\,. \quad (24)$$

Such an expansion may not be validated unless the action variable H is properly chosen. Since the orbit has different topology depending on the presence of transition point, the choice of the variable H has to be made separately for each case.

4.1 Orbit With Transition Point

Suppose the orbit curve $J(\psi,\theta;\lambda)$ = const. crosses the curve $\lambda B_{\max}(\psi,\theta) = 1$ at the points $(\psi_t \pm\theta_t)$, which are called as transition points. At the points, the distribution function of ripple-trapped particle is equal to that of passing particles. If we choose as

$$H = \psi_t(J,\lambda). \quad (25)$$

Then we have

$$F_0^{(0)}(H) = f_0(\psi_t)\,. \quad (26)$$

The correction $F_1^{(0)}$ is determined by eq. (23), where

$$\Delta F = -f_0\{h(\theta_t) - h(-\theta_t)\}. \quad (27)$$

4.2 Orbit Without Transition Point

When the collisionless orbit has no transition point, the distribution function is periodic with respect to ϑ; i.e. $\Delta F = 0$. Such orbit may exists in both sides of the orbit with transition points. We choose H so that the coefficients a_2 and a_1 are small enough. For instance, we choose such that

$$\left.\frac{\mathrm{d}J}{\mathrm{d}\lambda}\right|_H \equiv -\frac{\left.\frac{\partial H}{\partial \lambda}\right|_J}{\left.\frac{\partial H}{\partial J}\right|_\lambda} = \frac{\left\langle vM\frac{\partial J}{\partial \lambda}\right\rangle_\vartheta}{\langle vM\rangle_\vartheta} \qquad (28)$$

If the curve H = constant crosses the boundary curve with the region with transition points, the function $F_0^{(0)}$ is determined from the continuity condition at the boundary.

The correction $F_1^{(0)}$ is determined from eq. (23) with proper boundary conditions.

5. Transport Flux in Collisionless Regime

The transport flux for the mono-energetic particles is obtained by the integral

$$\Gamma = \int \mathrm{d}\theta \int \mathrm{d}\lambda F^{(1)} \frac{1}{e}\frac{\partial J}{\partial \theta}, \qquad (29)$$

where $F^{(1)}$ is solved from eq. (16). The integral is carried over the fixed magnetic surface. If we consider the case that the helical ripple is small, the part of the ripple-trapped particle with transition gives the main contribution for the transport flux.

Since the right hand side of eq. (16) is expressed by the linear combination

$$\mathscr{L} F^{(0)} \equiv L_1 \frac{\partial f_0}{\partial H} + L_2 \frac{\partial^2 f_0}{\partial H^2}, \qquad (30)$$

we can also express $F^{(1)}$ in the form

$$F^{(1)} \equiv \mathscr{G}_1(H,\vartheta,\lambda)\frac{\partial f_0}{\partial H} + \mathscr{G}_2(H,\vartheta,\lambda)\frac{\partial^2 f_0}{\partial H^2} \quad (31)$$

where $\mathscr{G}_1$ and $\mathscr{G}_2$ are obtained by solving the equation

$$K(J,\lambda)\frac{\partial \mathscr{G}_j}{\partial \vartheta} = L_j \qquad (j=1,2). \qquad (32)$$

The transport flux across the magnetic flux can be obtained as

$$\Gamma = \int \mathrm{d}\theta \int \mathrm{d}\lambda\, \mathscr{G}_1(H,\vartheta,\lambda)\frac{1}{e}\frac{\partial J}{\partial \theta}\frac{\partial f_0}{\partial H} + \int \mathrm{d}\theta \int \mathrm{d}\lambda\, \mathscr{G}_2(H,\vartheta,\lambda)\frac{1}{e}\frac{\partial J}{\partial \theta}\frac{\partial^2 f_0}{\partial H^2}. \qquad (33)$$

The integration with respect to θ is transformed into the integral with respect to H, which is expressed in terms of ψ. Thus, we obtain the expression

$$\Gamma = \int_{\psi_L}^{\psi} \mathrm{d}\psi \mathscr{V}_1(\psi,\psi_*)\frac{\partial f_0}{\partial \psi_*} + \int_{\psi_L}^{\psi} \mathrm{d}\psi \mathscr{V}_2(\psi,\psi_*)\frac{\partial^2 f_0}{\partial \psi_*^2}, \qquad (34)$$

where

$$\mathscr{V}_j(\psi,H) \equiv \frac{1}{e}\int \mathrm{d}\lambda\, \mathscr{G}_j \left.\frac{\partial J}{\partial H}\right|_{\psi,\lambda} \qquad (35)$$

for j = 1, 2. Carrying out the integration of the second term by parts, we can write

$$\Gamma = \mathscr{V}_2(\psi,\psi)\frac{\partial f_0}{\partial \psi} + \int_{\psi_L}^{\psi} \mathrm{d}\psi \mathscr{V}(\psi,\psi_*)\frac{\partial f_0}{\partial \psi_*}. \qquad (36)$$

The transport flux cannot be expressed by single coefficient. Since L_j is proportional to the collision frequency, the obtained flux is proportional to the collisionality.

6. Summary

Even in the low collisionality case, the distribution function cannot be expanded in power series of collision frequency. In the part of the phase space where the two orbits with different topology are proximate the boundary layer appears.

The distribution function of the passing particles is essentially local Maxwellian with small deviation. The part of distribution function of the ripple-trapped particles is determined by the transition point. In the collisionality regime ($\omega_T \ll \nu_{eff} \ll \iota v_T / R$) the local diffusion coefficient inversely proportional to collision frequency is obtained. In the very low collisionality regime ($\nu_{eff} \ll \omega_T$), the transport flux has nonlocal expression, proportional to the collisionality. In the collisionality between these two limits the analytic expression for the transport flux cannot be obtained.

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