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Studies on Modes in Weakly Relativistic Magnetized Plasmas

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Abstract

Considering magnetized homogeneous and inhomogeneous plasmas with weakly relativistic ions and electrons we derive the dispersion relations by carrying out the usual mode analysis of the ion and electron fluid equations. In addition, the reductive perturbation technique is employed to derive the expressions for the phase velocities of the modes by using a small dimensionless expansion parameter ε for the dependent quantities. Three types of modes in inhomogeneous plasmas and two types of modes in homogeneous plasmas are found to occur and usual mode analysis confirms their acoustic nature. In homogeneous plasmas, the possibility of instability is predicted on the basis of mode analysis. The conditions on modes propagation in the plasmas are achieved and the effects of various parameters on their phase velocities are analyzed. It is suggested that all these modes should be taken into account for the studies on solitons.

Keywords:

magnetized plasma, relativistic ions, relativistic electrons, modes, dispersion relation, reductive perturbation technique, phase velocity

1. Introduction

There are a variety of equations that govern the evolution of nonlinear phenomena in plasmas. The nonlinear phenomena that may be observed in higher-regions of the near-Earth plasma can be of different kinds e.g. heating type, parametric type etc. and another important class of such phenomena is solitons [1]. Production and propagation of ion acoustic solitons were studied by Ikezi [2], and Dahiya *et al.* [3] observed their partial reflection from a negatively biased grid immersed in the plasma. Later, Nishida [4] made extensive studies on the soliton reflection from a planar metallic plate, glass plate and metallic mesh.

For a weakly nonlinear and dispersive plasma, a time-dependent perturbation leads to the well-known Korteweg-deVries (K-dV) equation which describes one-dimensional solitary waves (solitons). Using K-dV equation, many researchers have studied the ion acoustic solitons in different plasma systems including the relativistic plasmas [5-10] which can be realized in the plasma sheet boundary layer of the Earth's magnetosphere, solar wind, solar flare and under the influence of high-power laser radiation. However, most of the studies on solitons are limited to some particular mode (mainly fast mode), whereas other modes may also occur in the plasmas under the effect of magnetic field and can correspond to the solitons of different behaviour. Therefore, our main concern in the present paper is to study the different types of modes in relativistic magnetized plasmas in view of soliton propagation.

2. Basic Formulation: Fluid Equations

We consider a magnetized and spatially inhomogeneous plasma having weakly relativistic ions and

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electrons. The ratio of particle pressure to magnetic field pressure (β) is taken to be small so that the diamagnetic effect is small and a uniform external magnetic field $\boldsymbol{B}$ ($= B_0 z$) along z-axis is assumed. The wave propagation is considered to be in the (x, z) plane. Under these conditions, the following basic equations are obtained for the ion and electron fluids.

$$\partial n_j/\partial t + \nabla\bullet(n_j \boldsymbol{v}_j) = 0\ , \tag{1}$$

$$n_j m_j \mathrm{d}(\gamma_j \boldsymbol{v}_j)/\mathrm{dt} = -s\, n_j e \nabla\phi + s\, n_j e(\boldsymbol{v}_j \times \boldsymbol{B}) - C_j T_j \nabla n_j\ , \tag{2}$$

$$-\varepsilon_0 \nabla^2 \phi = e(n_\mathrm{i} - n_\mathrm{e})\ . \tag{3}$$

Here, eq. (1) is the continuity equation and eq. (2) is the momentum equation for ion (j = i and s = +) and electron (j = e and s = –) fluids, with n_j as the density, m_j as the mass, $\boldsymbol{v}_j$ as the fluid velocity, $\gamma_j = (1 - v_j^2/c^2)^{-1/2}$ as the relativistic factor, C_j as the specific heat ratio and T_j as the kinetic temperature. Equation (3) is the Poisson's equation with ϕ as the electrostatic potential. It can be noted that the fluid equations for electrons with relativistic effect have been introduced in place of their Boltzmann distribution, since the wave velocity in the relativistic plasma can be comparable to the electron thermal velocity [7].

Now we carry out the usual mode analysis of eqs. (1)–(3) to find the dispersion relation for linear waves. The wave propagation is taken to be almost perpendicular to the direction of magnetic field (z-axis) so that the ions and electrons can preserve the charge neutrality. We assume all the perturbations of the form of $\exp[i(kx - \omega t)]$ in (x, z) plane. Under these conditions, we find the following equation when the wave width is taken to be smaller in comparison to the scale length of density inhomogeneity ($\partial n_0/n_0\partial x$, $z \ll k$)

$$1 - \Sigma_{j=\mathrm{i,e}}\, \omega_{\mathrm{p}j}^2/\{\gamma_j(\omega - kv_j)^2 - \omega\Omega_j^2/(\omega - kv_j) - k^2 C_j T_j/m_j - [se\Omega_j^2/n_0 m_j(\omega - kv_j)^2] n_{0z}\phi_{0z}\} = 0\ . \tag{4}$$

In eq. (4), n_0 is the unperturbed plasma density and $\omega_{\mathrm{p}j}$ and Ω_j are the plasma frequency and cyclotron frequency of the jth species, respectively. $v_j = u$ (v) is the initial ion (electron) velocity in the x-direction and n_{0z} and ϕ_{0z} are the density and potential gradients in the z-direction, respectively.

Solution of eq. (4) will give the relation between ω and k for the present case where ions as well electrons are taken to be weakly relativistic in an inhomogeneous magnetized plasma. If we consider a simplified case where ions and electrons are nonrelativistic ($v_j = 0$ and $\gamma_j = 1$) and the plasma is homogeneous ($n_{0z} = 0$ and $\phi_{0z} = 0$) and unmagnetized ($\Omega_j = 0$), the above equation assumes the following form

$$1 - \omega_{\mathrm{p}j}^2/[\omega^2 - k^2 C_\mathrm{i} T_\mathrm{i}/m_\mathrm{i}] - \omega_\mathrm{pe}^2/[\omega^2 - k^2 C_\mathrm{e} T_\mathrm{e}/m_\mathrm{e}] = 0\ . \tag{5a}$$

If the electron mass m_e is neglected in eq. (5a), the following relation between ω and k is obtained

$$\omega/k = [C_\mathrm{i} T_\mathrm{i}/m_\mathrm{i} + C_\mathrm{e} T_\mathrm{e}/m_\mathrm{i}(1 + k^2\lambda_\mathrm{De}^2)]^{1/2}\ . \tag{5b}$$

This relation is same as obtained by Chen [eq. (4–48) of ref. [11]] for ion waves in a plasma when the Poisson's equation is taken into account and the neutrality condition $n_\mathrm{i} = n_\mathrm{e}$ is not used, except the appearance of specific heat ratio C_e which he took unity for isothermal electrons. This relation is a dispersion relation for ion waves showing their acoustic nature where the inertia is provided by the ion mass and the pressure by the electron temperature. However, the contribution of finite electron mass to these ion acoustic waves in the plasma is reflected through eq. (5a). Moreover, since eq. (4) is a fourth-order equation in ω and k, it will give four dispersion relations with due corrections of the effects of finite electron mass, relativistic speeds, inhomogeneity and magnetic field. Subsequently, it is expected that four types of modes will appear in the present plasma model.

Now to find the phase velocity relations of the expected modes, we employ the reductive perturbation technique of normalized basic fluid equations separately for the homogeneous and inhomogeneous plasmas along with the same normalization as used in ref. [10].

3(a). Homogeneous Plasma

For the homogeneous medium, the following stretched co-ordinate system is introduced

$$\xi = \varepsilon^{1/2}(\boldsymbol{k}\bullet\boldsymbol{r} - \lambda t) = \varepsilon^{1/2}(x\sin\theta + z\cos\theta - \lambda t)\ ,\quad \eta = \varepsilon^{3/2} t\ . \tag{6}$$

Here, $\boldsymbol{k}$ is the unit vector along the direction of the wave propagation that makes an angle θ with the direction of the magnetic field. λ is the phase velocity of ion acoustic wave in (ξ, η) space and ε is a small dimensionless expansion parameter.

The quantities n, n_e, ϕ, u_x, u_y, u_z, v_x, v_y and v_z can be expanded around the equilibrium state in terms of ε as below

$$\begin{aligned} f &= 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots\,, \\ g &= \varepsilon g_1 + \varepsilon^2 g_2 + \dots\,, \\ h &= \varepsilon^{3/2} h_1 + \varepsilon^2 h_2 + \dots\,, \\ p_j &= p + \varepsilon^{3/2} p_{j1} + \varepsilon^2 p_{j2} + \dots\,, \end{aligned} \tag{7}$$

where $f \equiv (n, n_e)$, $g \equiv (\phi, u_z, v_z)$, $h \equiv (u_y, v_y)$ and $p_j \equiv (u_x, v_x)$ along with $p = u$ or v.

Now the normalized form of the basic fluid eqs. (1–3) gives the following relations in first-order quantities at different orders of ε along with the use of eqs. (6) and (7).

$$\textit{At order } \varepsilon: \quad n_1 - n_{e1} = 0\,. \tag{8a}$$

At order $\varepsilon^{3/2}$:

$$-(\lambda - b\sin\theta) n_{j1\xi} + \cos\theta\, b_{z1\xi} = 0\,, \tag{8b}$$

$$sm\sin\theta\, \phi_{1\xi} - smAb_{y1} + s_g m \sin\theta\, n_{j1\xi} = 0\,, \tag{8c}$$

$$\begin{aligned} &-(\lambda - b\sin\theta) b_{z1\xi} + s_g m \cos\theta\, n_{j1\xi} \\ &+ sm\cos\theta\, \phi_{1\xi} = 0\,. \end{aligned} \tag{8d}$$

In the above equations, $b = u$ (v) is ion (electron) fluid velocity, $s = +$ $(-)$ for ion (electron) fluid, $m = 1$ (m_r) for ion (electron) fluid and $s_g = 2\sigma$ (1) for ion (electron) fluid, where $m_r = m_i/m_e$, $\sigma = T_i/T_e$ and $A = \Omega_i/\omega_{pi}$. The integration of eqs. (8) under the boundary conditions that $n, n_e \to 1$ and $\phi, u_y, u_z, v_y, v_z \to 0$ as $\xi \to \pm\infty$ yields the following phase velocity relation for the ion acoustic wave.

$$\begin{aligned} \lambda = \{&(u + vm_e/m_i)\sin\theta \\ &\pm [(1 + m_e/m_i)(1 + 2\sigma)\cos^2\theta \\ &- (m_e/m_i)(u - v)^2 \sin^2\theta]^{1/2}\}/(1 + m_e/m_i) \end{aligned} \tag{9}$$

It is evident from eq. (9) that two types of modes, namely fast mode (λ_F): corresponding to plus sign and slow mode (λ_S): corresponding to minus sign are possible in a magnetized homogeneous plasma having weakly relativistic ions and electrons and their phase velocities depend on u, v, σ and θ.

3(b). Inhomogeneous Plasma

Using the same stretched co-ordinates and the perturbation of dependent quantities as taken in ref. [10], we can obtain the following relations at different orders of ε.

$$\textit{At order } \varepsilon^{1/2}: \quad n_{0\xi} = \lambda_{0\xi} = \phi_{0\xi} = 0\,. \tag{10a}$$

This equation indicates that the zeroth-order (unperturbed) quantities vary only with space not with time, implying the inhomogeneity present in the plasma.

$$\textit{At order } \varepsilon: \quad n_1 = n_{e1}\,, \tag{10b}$$

At order $\varepsilon^{3/2}$:

$$\begin{aligned} &-(\lambda_0 - b\sin\theta) n_{j1\xi}/\lambda_0 \\ &+ n_0\cos\theta\, b_{z1\xi}/\lambda_0 + b\sin\theta\, n_{0\eta} = 0\,, \end{aligned} \tag{10c}$$

$$\begin{aligned} b_{y1} = &\sin\theta\, \phi_{1\xi}/A\lambda_0 + ss_g\sin\theta\, n_{j1\xi}/An_0\lambda_0 \\ &+ \sin\theta\, \phi_{0\eta}/A + ss_g\sin\theta\, n_{0\eta}/An_0\,, \end{aligned} \tag{10d}$$

$$\begin{aligned} &-n_0(\lambda_0 - b\sin\theta) b_{z1\xi}/\lambda_0 + s_g m\cos\theta\, n_{j1\xi}/\lambda_0 \\ &+ smn_0\cos\theta\, \phi_{j1\xi}/\lambda_0 + s_g m\cos\theta\, n_{0\eta} \\ &+ smn_0\cos\theta\, \phi_{0\eta} = 0\,, \end{aligned} \tag{10e}$$

$$\textit{At order } \varepsilon^2: \quad n_0 \sin\theta\, b_{x1\xi}/\lambda_0 = 0\,. \tag{10f}$$

When we solve eqs. (10) for the first-order quantity n_1, we obtain

$$\begin{aligned} n_1 = \{&\xi[m\cos\theta + (\lambda_0 - v\sin\theta)v\sin\theta/\cos\theta \\ &+ 2\sigma m\cos\theta + (\lambda_0 - u\sin\theta)mu\sin\theta/\cos\theta] \\ &\times n_{0\eta}\}/[m\cos\theta/\lambda_0 - (\lambda_0 - v\sin\theta)^2/\lambda_0\cos\theta \\ &+ 2\sigma m\cos\theta/\lambda_0 - m(\lambda_0 - u\sin\theta)^2/\lambda_0\cos\theta]\,. \end{aligned}$$

Here, it can be noted that the left hand side is a first-order term and the right hand side contains only the zeroth-order terms. Since a first-order term cannot be explicitly expressed in terms of only the zeroth-order terms, the right hand side is made indeterminate by putting separately the numerator and denominator equal to zero. This process yields three relations for the phase velocity λ_0, out of which the two relations: corresponding to the denominator, are the same as obtained in case of homogeneous plasma [eq. (9)]. The third mode: corresponding to the numerator, has the following phase velocity relation

$$\begin{aligned} \lambda_T = [&(u^2 + v^2 m_e/m_i)\sin^2\theta \\ &- (1 + 2\sigma)\cos^2\theta]/(u + vm_e/m_i)\sin\theta\,. \end{aligned} \tag{11}$$

4. Results and Discussion: Propagation of Modes

To examine the propagation of modes, we analyze the phase velocity relations (9) and (11). Relation (9) shows that the phase velocity of the fast mode (λ_F) and slow mode (λ_S) will be real if the quantity appearing in the square root term be positive, which reads

$$\tan^2\theta \leq (1 + m_i/m_e)(1 + 2\sigma)/(u - v)^2 . \qquad (12)$$

For the slow mode, the phase velocity λ_S will be positive if the first term of R.H.S. of eq. (9) be greater than the second one. This requirement leads to the following condition

$$\tan^2\theta \geq (1 + 2\sigma)/(u^2 + v^2 m_e/m_i) . \qquad (13)$$

The third type of mode shall propagate in the plasma if the first term of eq. (11) be greater than the second one. This yields the same limiting condition on the wave propagation angle θ, as given by eq. (13). Therefore, we conclude the following on the basis of above inequalities

1. For the propagation of the fast mode, the wave propagation angle θ has an upper limit, given by eq. (12). For example, when $\sigma = 0.05$, $u = 45$ (70) and $v = 55$ the upper limit of the angle is 77.458° (71.546°). This limiting value is decreased for the higher ion speeds and lower ion temperatures.

2. The slow mode can propagate only for the certain range of angle θ prescribed by eqs. (12) and (13). For example, when $\sigma = 0.05$, $u = 45$ (70) and $v = 55$, the range is $1.330° < \theta \leq 77.458°$ ($0.858° < \theta \leq 71.546°$). This range of θ is decreased for the higher ion speeds.

3. The third type of mode propagates only when the wave propagation angle θ exceeds the lower limit given by eq. (13). For above set of values, this lower limit is 1.33° for ion streaming speed $u = 45$ and it is decreased to 0.858° when u is increased to 70.

Now we study the effect of ion and electron streaming speeds, ion to electron temperature ratio (σ) and the wave propagation angle θ on the phase velocities of the above three types of the modes. Figure 1 shows the opposite behaviour of the phase velocities of fast and slow modes with electron streaming speed v. Here, it is clear that the phase velocities have weak dependence on the electron speed, which is caused by the electron mass m_e appearing together with v in the phase velocity relations. It is evident from the figure that λ_F (λ_S) attains maximum (minimum) value when the electron speed becomes almost equal to that of the ions. This may be attributed to the energy exchange mechanism with the modes; however, one would have to take up kinetic approach for the present plasma model to examine such mechanism. For the third type of mode also, it is seen that λ_T increases slowly from 27.4681 to 27.4752 when v is raised from 45 to 70. If we analyze the effect of ion speed, the numerical calculations show that λ_F increases rapidly from 23.414 to 35.906 and λ_S from 21.614 to 34.123 and λ_T from 22.466 to 34.973 when u is raised from 45 to 70. It can be noted here that the magnitudes of λ_F, λ_S and λ_T follow $\lambda_F > \lambda_T > \lambda_S$.

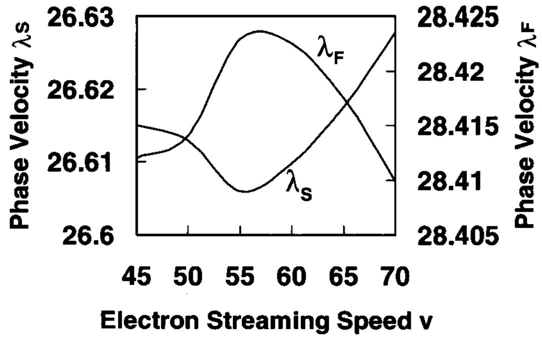

Fig. 1 Weak dependence of the phase velocities λ_F and λ_S of fast mode and slow mode, respectively, on the electron streaming speed v. Here, $\sigma = 0.05$, $\theta = 30°$ and $u = 55$.

Strong dependence of the phase velocities of all the modes on wave-propagation angle θ (in degrees) is portrayed in Fig. 2. This figure shows that the phase velocities of the modes get higher for the larger wave-propagation angles and the magnitudes of the velocities follow the same trend i.e. $\lambda_F > \lambda_T > \lambda_S$. Further, it is easy to examine the effect of ion temperature (σ) on λ_F, λ_S and λ_T through eqs. (9) and (11). For higher ion temperatures, the phase velocity of the fast mode increases and those of the slow and third type of modes get lower.

Now we analyze the results of reductive perturbation technique and those of normal mode analysis. Normal mode analysis shows the presence of four types of modes in the present plasma model. Since two real dispersion relations are inferred by the reductive perturbation technique for the homogeneous plasma, the other two roots may be complex and

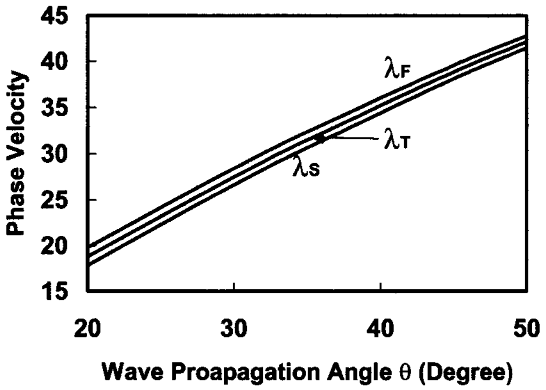

Fig. 2 Variation of the phase velocities λ_F, λ_S and λ_T of fast mode, slow mode and third type of mode, respectively, with the wave propagation angle θ. Here, $\sigma = 0.05$ and $u = v = 55$.

therefore instability may be expected in such a plasma model. However, in inhomogeneous plasma, we cannot expect the instability because three real dispersion relations are obtained. Further, it is suggested that all the modes should be taken into account while studying the solitons in magnetized plasmas having weakly relativistic ions and electrons, because they may correspond to the solitons of different behaviour.

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