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Nonlinear Stationary Structures in a Drfit Wave Turbulence Modulated by Zonal Flows

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Abstract

We investigate nonlinear stationary structures in a system of coupled equations describing drift wave turbulence and associated self-consistent zonal flows. The short scale drift wave turbulence is described by a kinetic wave equation for the action density of drift waves, whereas the longer scale zonal flows are described by a dynamic equation for the toroidally and poloidally symmetric component of the potential. A variety of radially propagating coherent structures such as nonlinear periodic wave trains, solitons and shocks in the modulation envelope are shown to be possible.

Keywords:
drift wave, zonal flow, coherent structure, stationary structure, turbulence

1. Introduction

Recently, there has been a resurgence of interest in the study of zonal flows generated by drift wave turbulence in toroidal magnetically confined plasmas [1-4]. Zonal flows are radially structured poloidal flows with toroidal and poloidal symmetry ($m = n = 0$), which are generated by a modulational instability mechanism operating on the turbulence. Physically, energy is pumped into longer scales because of an inverse cascade in the turbulence and $m = n = 0$ perturbations are preferred because they are unaffected by magnetic shear in the confinement device. The sheared zonal flows are typically randomly oriented and induce a strong refractive effect on the propagating drift waves. In the weakly turbulent limit, this random shearing refraction leads to a quasilinear diffusive spreading of the drift wave turbulence spectrum in the radial wavenumber space, finally producing a nonlinear stationary state with random phased, near Gaussian, turbulence. This limit has been extensively studied in recent years [4] and has led to the conclusion that the saturated drift wave fluctuation level is quenched by zonal flow effects to amplitudes well below the typical mixing length estimates $(e\tilde{\phi}_M/T) \sim (\rho_s/L_n)$($\rho_s$ is the ion Larmor radius at electron temperature and is a measure of typical scale-length of turbulence and L_n is the scale-length of density gradient driving the turbulence and subscript M refers to mixing length saturation level), and to avalue determined by the neoclassical damping rate ν of zonal flows [5], viz. $(e\tilde{\phi}_S/T) \sim (\rho_s/L_n)(\nu/\omega_*)^{1/2}$ (where $\nu/\omega_* < 1$, ω_* being the typical drift wave frequency and subscript S on $\tilde{\phi}$ refers to new saturation level). The associated anomalous transport coefficients also show the above improvement. It is easy to understand the improvement on physical grounds. As the damping parameter ν drops, the zonal flows become stronger and limit the turbulent fluctuations to lower amplitudes. An obvious question arises as to what happens as the damping parameter $\nu \to 0$. As the zonal flows get stronger and stronger, the assumption of a weakly turbulent state and quaslinear diffusion of the drift wave turbulence spectrum become questionable. Instead, we enter a regime of strong turbulence where coherent

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zonal flow structures are formed and produce a 'trapping' of significant numbers of drift wave trajectories; in their turn, collective Reynold stresses due to 'trapped' and 'untrapped' drift wave distributions sustain the zonal flow. These coupled nonlinear structures are similar in nature to classical Bernstein - Greene - Kruskal waves which have been extensively studied for Vlasov - Poisson set of equations in one-dimensional electrostatic plasmas [6] and have also been extended to wave-wave interaction phenomena [7].

In this letter we investigate nonlinear stationary structures in a system of coupled equations descrbing drift wave turbulence and associated self-consistent zonal flows. A variety of radially propagating coherent structures such as nonlinear periodic wave trains, solitons and shocks in the modulation envelope are shown to be possible. We expect such structures to be relevant for interpretation of observed non-Gaussian features of the turbulence such as coherence, intermittency, bursty transport [8] etc. Many of the features of drift wave turbulence studies here should also be important for Rossby waves in the atmosphere, since the fundamental equations describing these two varieties of turbulence are very similar [2,9].

2. Nonlinear Theory

We look for nonlinear stationary solutions for the coupled drift-wave zonal flow problems in the absence of sources and sinks for drift waves ($\gamma_k = \Delta\omega_k \equiv 0$). Going to amoving frame propagating with radial velocity U and assuming stationarity in that frame, we may replace $\partial/\partial t$ by $-U\partial/\partial x$ in the kinetic wave equation for dimensionless action density $N_k = |e\varphi_k/T|^2 (1 + k^2\rho_s^2)^2$ of drift waves and nonlinear equation for the zonal flow potential $\psi = e\phi/T_e$ to obtain the set:

$$(v_{gx} - U)\frac{\partial N_k}{\partial x} + k_y \frac{\partial}{\partial x}\left(v + \frac{\partial^2 v}{\partial x^2}\right)\frac{\partial N_k}{\partial k_x} = 0 \qquad (1)$$

$$\left(\mu\frac{\partial^2}{\partial x^2} - v\right)v + U\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\sum_k \frac{k_x k_y}{\left(1+k^2\right)^2} N_k \qquad (2)$$

where $v \equiv d\psi/dx$ is the magnitude of zonal flow velocity. We have introduced the normalizations: x to ρ_s, k to ρ_s^{-1}, v to drift speed $= v_* \equiv \rho_s/L_n)c_s$, t to ρ_s/v_*, N_k to the mixing length estimate $(\rho_s/L_n)^2$ and μ, v to c_s/L_n. v_{gx} is the radial group velocity of drift waves given by the normalized expression $-2k_xk_y/(1 + k^2)^2$. Eq. (1) conserves N_k along the characteristics given by

$$\frac{dk_x}{dx} = k_y \frac{d}{dx}\left(v + \frac{d^2 v}{dx^2}\right)/(v_{gx} - U), \quad \frac{dk_y}{dx} = 0 \qquad (3)$$

Eq. (3) may be readily integrated to give the constants of motion

$$W = v + \frac{d^2 v}{dx^2} - \frac{1}{1+k^2} + \frac{k_x}{k_y}U;\, k_y = \text{Constant} \qquad (4)$$

and suggests an exact solution to eq. (1)

$$N_k(k_x, k_y, x) \equiv N_k(W(k_x, x), k_y)\,. \qquad (5)$$

k_yW may be physically interpreted as the frequency of the drift wave as seen from a frame moving radially with velocity U, corrected for effects due to velocity and density perturbations - $(v + d^2v/dx^2)$ terms - associated with the zonal flow; this frequency is an invariant (a constant of motion) because in the moving frame, the zonal flow structures are independent of time. We thus expect, that as the drift wave encounters regions with different $v(x)$ and $v''(x)$ (prime denotes an x derivative), its k_x will change in such a way as to keep the wave number k_y and the frequency (k_yW) constant. When k_x, $k_y \ll 1$, we may rewrite the constant of motion W as $W \simeq K_x^2 + f(x)$ where $K_x = k_x + U/2k_y$ and $f(x) = v + v'' + k_y^2 - (1 + U^2/4k_y^2)$. This is of the standard form of a kinetic energy term (K_x^2) and a potential energy term $(f(x))$ combining to give a constant of motion. This equation indicates that if f has a maximum value f_m, we must distinguish between 'trapped' drift waves with $W < f_m$ (which sample only parts of x-space because in other parts $K_x^2 < 0$ and 'untrapped' drift waves with $W > f_m$ (which sample all of x-spacebecause $K_x^2 > 0$ everywhere). The expansion of $k^2 \ll 1$ used above is only a simplification to make the algebra less tedious and is no way necessary physically for the trapping of drift waves. The physical process responsible for trapping of drift waves near the minimum of a velocity shear layer may be readily studied by writing the characteristic ray equations for drift wave trajectories viz. (in dimensionless variables), $dx/dt = -2k_xk_y/(1 + k^2)^2$ and $dk_x/dt = k_y[\,(dv/dx) + (d^3v/dx^3)]$ and $dk_y/dt = 0$. Taking v near a minimum as $v = v_m + v''_m(x - x_m)^2/2$ we get the equation $(d^2k_x/dt^2) + [k_y^2 v''_m/(1 + k^2)^2]k_x = 0$. For $v''_m > 0$, this is the equation for a nonlinear oscillator as may be readily verified by multiplying the equation with dk_x/dt and integrating once. The solution may be described in terms of complete elliptic integral of the second kind

$$E\left(\cos^{-1} k_x/k_o, k_o/(1+k_y^2+k_o^2)^2\right) = \left(2k_y^2 v_m''\right)^{1/2} t/(1+k_y^2+k_o^2) \quad (6)$$

where we have used the boundary condition, $k_x = k_o$, $\partial k_x/\partial = 0$ at $t = 0$. When k_o^2, $k_y^2 \ll 1$, the solution may be written as

$$k_x = k_o cn\left[(2K/\pi)^2\left\{2k_y^2 v_m''/(1+k_y^2+k_o^2)^2\right\}^{1/2} t\right] \approx k_o \cos\left(\sqrt{2k_y^2 v_m''}\, t\right) \quad (7)$$

where K denotes the complete elliptic integral of first kind. Thus k_x and x are both periodic functions of t which become simple trignometric functions in the limit k_x, $k_y \ll 1$. Physically, as the drift wave creeps up the velocity shear layer from its minimum, it loses its k_x (x-momentum) in the process of satisfying the WKB conditions, the ray eq. (7). Finally k_x^2 hits zero and the wave gets reflected getting trapped in the trough, oscillating back and forth between the two reflection points with a typical bounce frequency ($\sqrt{2k_y^2 v_m''}$) (in terms of unnormalized variables $(c_s/L_n)(k_y\rho_s)(2\rho_s^2[d^2v_m/dx^2]/v_*)^{1/2}$) which is related to curvature of the shear layer near its minimum.

Following up on the discussion after eq. 2 we may now argue that solution for stationary eqs. (1) and (2) may be obtained by solving the nonlinear self-consistency condition:

$$\mu(d^2v/dx^2) - vv + U(dv/dx) = (-1/2)(d/dx) \left[\int_{-\infty}^{\infty} dk_y k_y \left\{\int_{f_m}^{\infty} dW J(W,k_y,f) N_U(W,k_y) + \int_f^{f_m} dW J(W,k_y,f) N_T(W,k_y)\right\}\right] \quad (8)$$

where $N_U(W, k_y)$ and $N_T(W, k_y)$, are the action densities for untrapped and trapped parts of the stationary drift wave turbulence, $J(W, k_y, f)$ is the Jacobian for transforming k_x variable to W and is given by the express

$$J = \left[(W-f)^{1/2} + U/2k_y\right] / \left\{(W-f)^{1/2}\left[1+k_y^2+\left\{(W-f)^{1/2}+U/2k_y\right\}^2\right]^2\right\} \quad (9)$$

The entire nonlinearity in eq. (8) comes from the dependence of the right side on $f = v + v'' + k_y^2 - 1 - v^2/4k_y^2$ which, in turn, is determined by the choice of the trapped and untrapped action densities $N_{U,T}(W, k_y)$. Since any function of the constants of motion W, k_y is an exact solution of eq. (1), there is an arbitrariness in the choice of $N_{U,T}$. This arbitrariness is well known in the standard BGK problem also. Ultimately, the choice of a suitable $N_{U,T}$ will be decided by conditions of accessibility as one follows up on an initial value problem and/or questions of stability of the nonlinear stationary states being discussed here. But these questions are beyond the scope of the present calculation and have to be treated separately. Here, to illustrate the nature of typical nonlinear structures, we shall make choices of N_T, N_U on intuitive grounds. We assume

$$N_U(W,k_y) = \frac{N_{oU}}{\left(1+\frac{(w-f_m)^2}{\Delta^2}\right)\left[1+\frac{(k_y-k_{yo})^2}{\delta^2}\right]^3} \qquad W > f_m$$

$$N_T(W,k_y) = \frac{N_{0T}(f_m-W)^{1/2}+N_{oU}}{\left[1+\frac{(k_y-k_{yo})^2}{\delta^2}\right]^3} \qquad f < W < f_m \quad (10)$$

The two distributions are chosen to be continuous at $f = f_m$ and the k_y dependence is chosen in such a manner that there is no symmetry in the $\pm y$ directions. This asymmetry reflects the fact that drift waves typically propagate in either the electron or ion diamagnetic drift direction. It also ensures that the required integrals do not vanish even when $U \to 0$. For large k_x, k_y the distributions drop-off as a power law which is typical of many saturated turbulent states. The action density for trapped waves is chosen in the so-called Bohm-Gross form with a maximum at the bottom of the trough and the phase space density dropping off to N_{oU} at the maximum as a square-root function; this is extensively used in the study of trapped particle effects in large amplitude electrostatic waves [7]. With this choice, we find

$$\mu v'' - vv + Uv' = (\alpha_1 - \alpha_o)(v' + v'') - \alpha_2\left[(v_m - v)^{3/2}\right]' - (3\alpha_o/2 - \alpha_1)\left[(v_m - v)^2\right]' \quad (11)$$

where, $\alpha_o \cong (3\pi^2 N_{oU}/16)k_{yo}\Delta\delta$, $\alpha_1 \cong (3\pi N_{oU}/16)k_{yo}\delta$ and $\alpha_2 \cong (\pi N_{oT}/8)k_{yo}\delta$. The coefficient α_o arises from the contribution of untrapped waves whereas the coefficients α_1 and α_2 arise from the contributions due to trapped waves. We now consider eq. (11) in various limits. Let us first ignore ν, μ, and the nonlinear terms. The linearised W may be fourier analyzed in x and solved to get the dispersion relation $U = (\alpha_1 - \alpha_o)(1 - q^2)$ where q is the normalized wave number for the zonal flow velocity v. Thus, coupling to trapped and untrapped drift waves converts the zonal flow perturbations into radially propagating dispersive waves. If we had retained, resonant wave-wave interactions, these dispersive waves would have exhibited the usual modulational instability. However, we are looking for stationary solutions where the 'resonant waves' with small k_x^2 have been trapped in the modulation envelope, thereby saturating the instability. The structure of the nonlinear stationary state is determined by the nonlinear trapped wave contribution $[\alpha_2(v_m - v)^{3/2}]'$ to eq. (11). Neglecting ν, μ and higher order nonlinearity term $[(v_m - v)^2]'$, eq. (11) may be rewritten as

$$V'' - K^2 V + \beta V^{3/2} = 0 \qquad (12)$$

where $V = v_m - v > 0$, $K^2 = [U \div (\alpha_1 - \alpha_o) - 1]$ and $\beta = \alpha_2/(\alpha_1 - \alpha_o)$. Equation (12) gives the quadrature $1/2(dv/dx)^2 + \Psi(V)$ = Constant where the Sagdeev pseudo-potential takes the form $\Psi(V) = -K^2\ V^2/2 + 2/5\beta^{5/2}$. This pseudopotential starts at $V = 0$ with zero slope, has a minimum, $-\Psi_m$, at $V = v_m = K^4/\beta^2$ and again takes on the value 0 at $V = V_o = 25/16\ K^4/\beta^2$. An effective particle starting at a $V > 0$ oscillates in the pseudopotential with a frequency which is a function of amplitude; the resulting $V(x)$ structure is a nonlinear periodic wave train with a period which is afunction of amplitude. As the initial V approaches zero, the amplitude of the nonlinear wave increases and so does its period. If V starts from zero, the period is infinite and we get a solitary pulse or a soliton. An exact solution may be written in this case, viz. $V = 25/16\ K^4/\beta^2\ sech^4\ [K/4(x - Ut)]$. Physically, this solution for $V \equiv v_m - v$ represents a pair of back to back velociy shear layers with a significant population of drift waves trapped between them and sustained collectively by the Reynold stresses due to background sea of untrapped and trapped drift waves. The nonlinear structure propagates radially as a soliton with nonlinearity and dispersion conspiring to give a width which goes inversly as the fourth root of the amplitude. If we retain the dissipative terms, the effective particle suffers a damped oscillation in the pseudopotential and finally settles down at the minimum value $v_m = K^4/\beta^2$. The corresponding $V(x)$ structure is a shock-like structure with V going from 0 to v_m after ringing a few times periodically around the final value. If the damping terms are large, there is no ringing and the shock solution goes monotonically from 0 to v_m. If the dominant dissipative term is viscous ($\nu = 0$, μ, large), the monotonic shock solution may be analytically written down in the form $V = K^4/\beta^2\{\exp(K^2(x - Ut)/\mu)/[1 + \exp(K^2(x - Ut)/2\mu)]^2\}$.

3. Conclusions

In conclusion, we have looked for stationary solutions describing nonlinear coherent structures in the coupled problem of drift wave turbulence and associated self-consistent zonal flows and find that when drift wave trappingis important, a variety of radially propagating structures such as periodic nonlinear wave trains, solitons and shocks in the modulation envelopes may be formed. These solutions represent alternate saturated states to modulation instabilities of drift wave turbulence - nonlinear states which are dominated by coherence and drift wave trapping in contrast to the usual saturated states dominated by random phases and quasilinear diffusive spreading of drift wave spectra. Such nonlinear coherent structures may have already been observed in the collisionless ($\nu = 0$) strongly turbulent drift wave imulation of Lin et al. [1] and may also be responsible for observed phenomena such as intermittency, oscillations and bursty transport in experiments on drift wave turbulence [8].

References

[1] G. Hammet *et al.*, Plasma Phys. Control. Fusion **35**, 973 (1993); Z. Lin, T.S. Hahm, W.M. Tang and R.B. White, Science **281**, 1835 (1998).

[2] A. Balk, S.V. Nazarenko and V.E. Zakharov, Sov. Phys. JETP **71**, 249 (1990).

[3] V.V. Lebedev, P.H. Diamond, V.D. Shapiro and G.I. Soloviev, Phys. Plasmas **2**, 4420 (1995).

[4] P.H. Diamond *et al.*, in *Proceedings of the 17th international conference on Plasma Physics and Controlled Nuclear Fusion Research*, Yokohama (International Atomic Energy Agency IAEA-CN-69/TH3/1, 1998); P.K. Kaw and R. Singh, Bull. Am. Phys. Soc. **44**, 1262 (1999); A. Smolyakov, *et al.*, Phys. Rev Lett. **84**, 491 (2000).

[5] B. Kim, P.H. Diamond, H. Biglari and J.D. Callen, Phys. Fluids B **3**, 384 (1991).

[6] I. Bernstein, J.M. Greene and M.D. Kruskal, Phys. Rev. **108**, 546 (1957).

[7] P. Kaw, K. Mima and K. Nishikawa, Phys. Rev.

Lett. **34**, 803 (1975).

[8] R. Jha *et al.*, Phys. Rev. Lett. **69**, 1375 (1992); Joseph *et al.*, Phys. Plasmas **4**, 4292 (1997); S. Benkadda *et al.*, Phys. Rev. Lett. **73**, 3403 (1994); B.A. Carreras *et al.*, Phys. Plasmas **3**, 2664 (1996).

[9] A. Hasegawa, C.G. Maclennen and Y. Kodama, Phys. Fluids **22**, 2122 (1979).

[10] V. Goldman, Phys. Fluids **13**, 1281 (1970); M.V. Goldman, Rev. Mod. Phys. **56**, 709 (1984).