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Nonlinear Waves along the Magnetic Field in a Multi-Ion-Species Plasma

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Abstract

Low-frequency waves propagating parallel to a uniform external magnetic field in a plasma containing multiple ion species are studied analytically and numerically. Theory shows that the presence of multiple ion species greatly changes properties of the left circularly polarized waves. The Alfvén and ion cyclotron waves are split into two modes in a two-ion-species plasma. These waves are unstable against modulational instability except for a small frequency domain. Their growth rates are especially large near the ion cyclotron frequencies. The evolution of the modulational instability is then investigated with simulations based on a three-fluid model. Theoretically obtained growth rates are found to be in good agreement with the simulation result.

Keywords:

Alfvén wave, ion cyclotron wave, multi-ion-species plasma, modulational instability, three-fluid model

1. Introduction

Astrophysical plasmas as well as fusion plasmas usually contain multiple ion species. In a typical astrophysical plasma, hydrogen (H) is the major component. The number density of helium (He) is about 10 % of that of hydrogen. The abundances of heavier ions such as C, O, and Fe are much smaller than that of He. Recently, propagation of magnetosonic waves in a multi-ion-species plasma has been extensively studied [1-6]. It has been then recognized that the presence of multiple ion species significantly change wave properties; linear and nonlinear wave propagation [1,2], particle acceleration [3,4], and dissipation of wave energy [5,6].

In those papers, perpendicular waves were investigated. In this paper we extend the work to the waves propagating along the external magnetic field, i.e., Alfvén, whistler, and ion cyclotron waves. It is found that multiple ion species have particularly significant effects on left circularly polarized waves, which will therefore be mainly described here.

2. Linear and Nonlinear Wave Theory

We discuss wave propagation on the basis of a fluid model with multiple ion species,

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \boldsymbol{v}_j) = 0 , \tag{1}$$

$$m_j \left(\frac{\partial}{\partial t} + (\boldsymbol{v}_j \cdot \nabla) \right) \boldsymbol{v}_j = q_j \boldsymbol{E} + \frac{q_j}{c} \boldsymbol{v}_j \times \boldsymbol{B} , \tag{2}$$

$$\frac{1}{c} \frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} , \tag{3}$$

$$\frac{1}{c} \frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times \boldsymbol{B} - (4\pi/c) \sum_j q_j n_j \boldsymbol{v}_j , \tag{4}$$

where the subscript j refers to ion species (species a, b, $\cdots$) or electrons (j = e); m_j is the mass, q_j the charge, n_j

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the number density, and $\boldsymbol{v}_j$ the velocity. We assume that the external magnetic field and wave propagation are both in the x direction ($\partial/\partial y = \partial/\partial z = 0$).

The linear dispersion relations can be obtained from the above set of equations as

$$c^2 k^2 = \omega^2 - \sum_j \omega_{pj}^2 \omega/(\omega \mp \Omega_j). \tag{5}$$

Here, Ω_j and ω_{pj} denote the cyclotron frequency and plasma frequency, respectively, $\Omega_j = q_j B_0/(m_j c)$ and $\omega_{pj} = (4\pi n_{j0} q_j^2/m_j)^{1/2}$; the subscript 0 refers to equilibrium values. The quantity Ω_j includes the sign of the charge q_j, and $\Omega_b < \Omega_a$ is assumed. The first term on the right-hand side of Eq. (5) arises from the displacement currents. In the following, their effects will be neglected.

The waves are circularly polarized. The electric field as well as magnetic and velocity perturbations has component perpendicular to the external magnetic field. There appears no parallel electric field in the linear theory. The minus and plus signs in the denominator on the right-hand side of Eq. (5) represent left (L) and right (R) circularly polarized waves, respectively. (Throughout this paper, the upper and lower signs represent the L and R waves, respectively.)

If we include kinetic pressure in Eq. (2), we have electrostatic waves. In the linear theory, however, they are decoupled from the electromagnetic waves given by Eq. (5).

We show in Fig. 1 dispersion curves of the L waves for a plasma containing hydrogen and helium ions with the density ratio $n_{He}/n_H = 0.1$; hence, a = H and b = He. The L waves have resonances at $\omega = \Omega_a$ and at $\omega = \Omega_b$, i.e., at ion cyclotron frequencies [7]. The high frequency mode has a cutoff frequency ω_0,

$$\omega_0 = \frac{c^2 |\Omega_e| \Omega_a \Omega_b}{v_A^2 (\omega_{pe}^2 + \omega_{pa}^2 + \omega_{pb}^2)}, \tag{6}$$

where v_A is the Alfvén speed, $v_A = B_0/(4\pi \sum_j n_{j0} m_j)^{1/2}$. In the frequency domain $\Omega_b < \omega < \omega_0$, the L waves cannot propagate.

On the other hand, the presence of heavy ions does not drastically change properties of the R waves; we will therefore restrict ourselves to the L waves.

For frequencies much lower than the ion cyclotron frequencies, Eq. (5) gives

$$\omega/k = v_A (1 \mp \mu v_A k), \tag{7}$$

where the constant μ is defined

$$\mu = \frac{1}{2} \sum_j \frac{\omega_{pj}^2}{\Omega_j^3} / \sum_j \frac{\omega_{pj}^2}{\Omega_j^2}. \tag{8}$$

In this low frequency regime, the dispersion relations are quite similar to the ones in a single-ion-species plasma. Hence, using a perturbation theory similar to that in refs. [8,9], we can derive derivative nonlinear Schrödinger equation for the low-frequency waves.

For frequencies of the order of ion cyclotron frequencies, the L waves have strong dispersion. It is expected then that these waves are described by the nonlinear Schrödinger equation [10-12]. To derive this type of equation, we intorduce stretched coordinates

$$\xi = \varepsilon (x - v_g t), \tag{9}$$

$$\tau = \varepsilon^2 t, \tag{10}$$

where ε is a smallness parameter and v_g is the group velocity, $v_g = \partial\omega/\partial k$. We expand transverse fields and velocities, i.e., their y and z components, as

$$B_y = \sum_{l=-\infty}^{\infty} \sum_{n=1}^{\infty} \varepsilon^2 B_{ynl} \exp[il(kx - \omega t)], \tag{11}$$

and longitudinal field E_x, velocities v_{jx}, and densities n_j as

$$E_x = \varepsilon^2 E_{x2} + \varepsilon^3 E_{x3} + \cdots. \tag{12}$$

Then, after some algebra, we obtain the nonlinear Schrödinger equation

$$i\frac{\partial \phi}{\partial \tau} + \beta \frac{\partial^2 \phi}{\partial \xi^2} + \alpha \left(|\phi|^2 - |\phi_0|^2 \right) \phi = 0. \tag{13}$$

Here, ϕ is defined as

$$\phi = (B_{y1} \pm i B_{z1})/B_0, \tag{14}$$

and ϕ_0 is the value of ϕ in the far upstream region; the coefficient β is $\beta = (1/2)\partial^2\omega/\partial k^2$, and α is given as

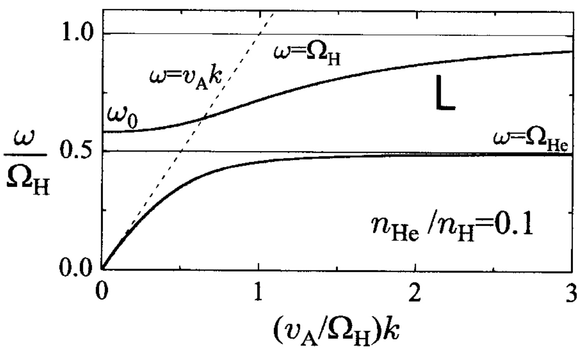

Fig. 1 Dispersion relations of the L waves in a H-He plasma with $n_{He}/n_H = 0.1$.

$$\alpha = -\frac{k}{4v_g}\left[-\frac{2\omega^2}{k^2}+\frac{4\omega v_g}{k}-v_g^2+\left(\frac{\omega^3}{k^2 v_g^2}-\frac{3\omega^2}{kv_g}\right)\frac{\partial v_g}{\partial k}\right.$$
$$\left.+\frac{\omega^3}{kv_g^3}\left(\frac{\partial v_g}{\partial k}\right)^2-\frac{\omega^3}{3kv_g^2}\frac{\partial^2 v_g}{\partial k^2}-\frac{c^2(kv_g+\omega)^2}{\sum_j \omega_{pj}^2}\right]. \quad (15)$$

(Equations (13)–(15) are also applicable to the R waves.)

We show in Fig. 2 coefficient α of the nonlinear term as a function of the frequency ω for a H-He plasma. For comparison, we also show, by the dotted line, the coefficient for a hydrogen plasma. The shaded areas indicate the frequency domains where the L waves cannot propagate. The coefficient α is always negative. It becomes quite large in magnitude near the resonance frequencies, $\omega \sim \Omega_{He}$ and $\omega \sim \Omega_H$, and near the cutoff frequency, $\omega \sim \omega_0$.

Coefficient β of the dispersion term is negative in the frequency domain $0 < \omega < \Omega_{He}$. It takes positive values in a small region just above the cutoff frequency ω_0. It is negative in a higher frequency region up to Ω_H.

The L waves are modulationally unstable when $\alpha\beta > 0$, and its growth rate Γ for a perturbation with a wave number K is given by [9,10]

$$\Gamma = |\beta K|\left(2\alpha|\phi_0|^2/\beta - K^2\right)^{1/2}. \quad (16)$$

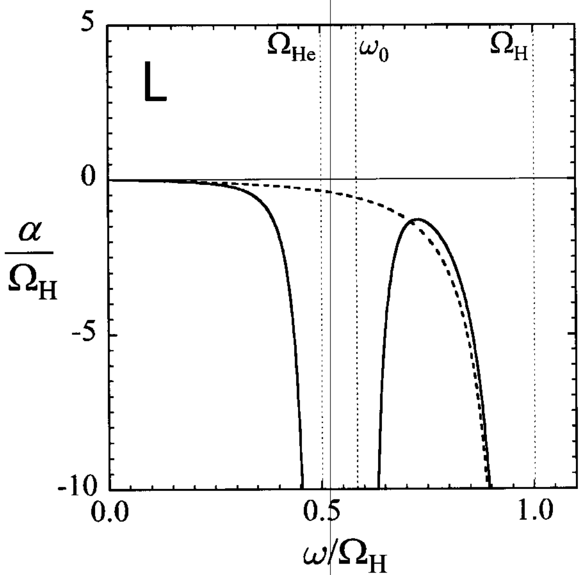

Fig. 2 Coefficient α of the nonlinear term as a function of the frequency ω. The solid and dotted lines show, respectively, cases for H-He plasma and H plasma.

3. Three-Fluid Simulation

To further study nonlinear evolution of the L waves in a two-ion-species plasma, we have carried out numerical simulations of the three-fluid model, basic equations of which were shown in Eqs. (1)–(4). We solved the initial value problem employing the pseudo-spectral method. We assume that the waves propagate in the x direction along the external magnetic field in a periodic system.

We have chosen the mass ratio between the hydrogen and electron as $m_H/m_e = 50$. The helium-to-hydrogen mass ratio is $m_{He}/m_H = 4$. The charge ratios are $q_H/q_e = -1$ and $q_{He}/q_H = 2$. In an equilibrium state, the density ratio is $n_{He}/n_H = 0.1$, and the magnetic field strength is $|\Omega_e|/\omega_{pe} = 0.5$, so that $c/v_A = 15.4$. The time step is $\omega_{pe}\Delta t = 0.2$. The number of grid points is 128. The grid spacing is $\Delta_g = 1.0\ c/\omega_{pe}$ for the wave with wave number $k = 1.6\ \Omega_H/v_A$.

First, we confirmed that the wave profiles given by the solitary wave solutions steadily propagate as the theory predicts. Then, we examined the modulational instability. As the initial wave profiles, we used periodic (sine or cosine) L waves obtained from the linearized three-fluid equations. We then observed their space-time evolution. We show in Fig. 3 profiles of B_y and $B_\perp$ at various times. Here, $B_\perp$ is defined as $B_\perp = B_y + iB_z$; thus, $|B_\perp|$ represents the envelope of the wave.

The wave number of the carrier wave is $kv_A = \Omega_H$ and the initial amplitude is $|B_\perp|/B_0 = 0.1$. The figure clearly indicates that the wave is unstable. Figure 4 shows a time variation of the amplitude of the modulation; $B_{\perp max} - B_{\perp 0}$, where $B_{\perp max}$ is the maximum value of $B_\perp$ and $B_{\perp 0}$ is the initial value. After the time $\Omega_H t \simeq 50$, the amplitude of the modulation keeps increasing. Its growth rate is $\Gamma/\Omega_H = 0.026$. In this way, we have observed the growth rates of the modulational instability for many different waves.

We show in Fig. 5 the growth rate as a function of the frequency. The solid lines represent the theory. The dots show simulation result. The growth rate becomes quite large for frequencies near Ω_{He} as well as near Ω_H. In the frequency region right above ω_0 (the shaded area with horizontal lines), the wave is stable, because $\alpha\beta < 0$ there. The theory and simulation are in good agreement.

4. Summary and Discussion

We have studied linear and nonlinear waves propagating along a uniform external magnetic field in a cold, multi-ion-species plasma; we have calculated

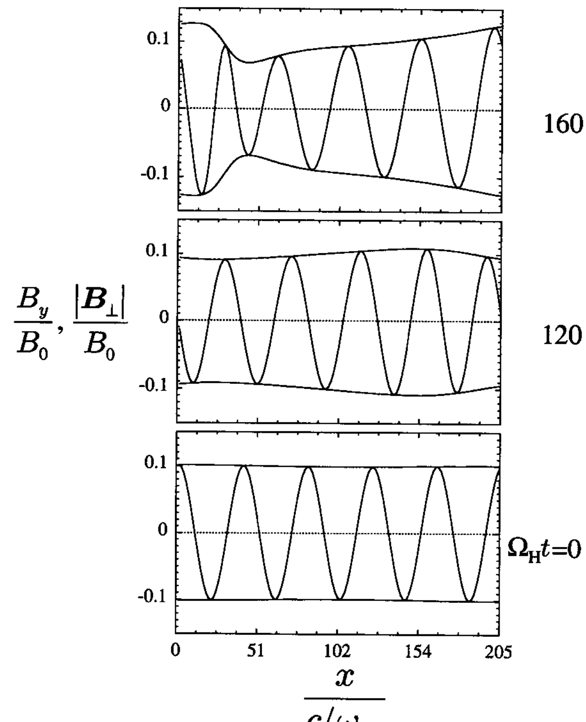

Fig. 3 Profiles of B_y and wave envelope $|B_\perp|$ at various times.

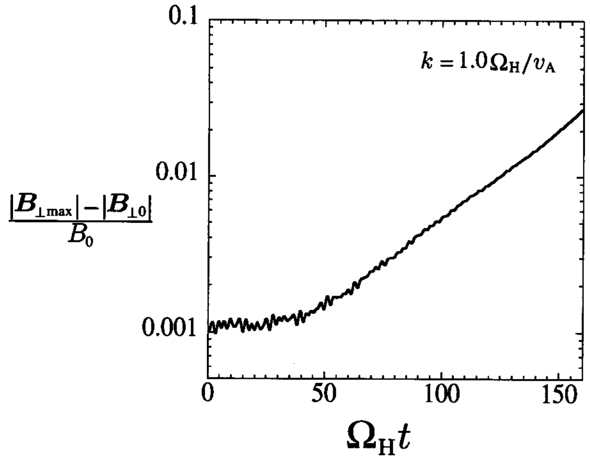

Fig. 4 Time variation of the perturbation of amplitude.

linear dispersion relations and derived derivative nonlinear Schrödinger equations for the low frequency waves and nonlinear Schrödinger equations for the waves with strong dispersion. Except for the frequencies near ω_0, the L waves are unstable against the modulational instability. The growth rates are especially great near the ion cyclotron frequencies. We then

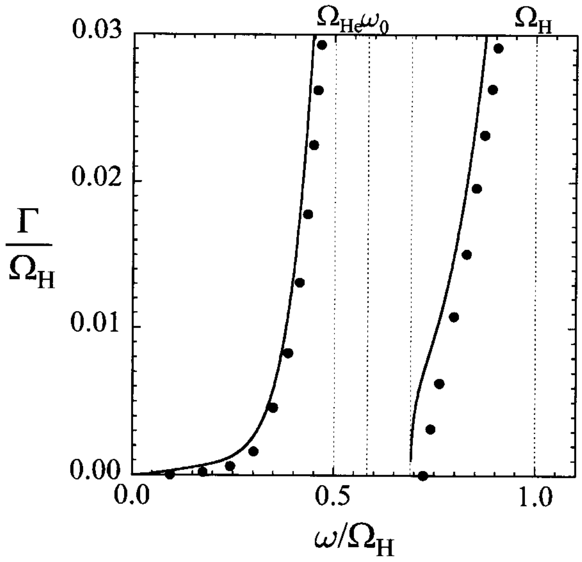

Fig. 5 Growth rate of modulational instability as a function of the frequency.

investigated nonlinear propagation of the L waves using a three-fluid simulation code. It was found that theoretically obtained growth rates for the modulational instability are in good agreement with the simulation result.

We have studied on the basis of the fluid model. Let us now discuss a kinetic effect; cyclotron damping. Calculations show that the damping rates of the L waves are extremely small for plasmas in magnetic tubes in the solar corona; it is one of our major motivations to study wave propagation in those plasmas. There, the temperature is very high and the density is low. Hence, the plasma is collisionless, and the main dissipation mechanism of the L waves is cyclotron damping.

We estimate the cyclotron damping rate. From a set of linearized Vlasov and Maxwell equations, we obtain kinetic dispersion relations for the waves propagating along a magnetic field as

$$\frac{k^2c^2}{\omega^2} = 1 - \sum_j \omega_{pj}^2 \left\{ \frac{1}{\omega(\omega-\Omega_j)} + \frac{k^2 v_{Tj}^2}{\omega(\omega-\Omega_j)^3} - \frac{i\sqrt{\pi}}{\sqrt{2}\,kv_T\,\omega} \exp\left[-\frac{1}{2}\left(\frac{\omega-\Omega_j}{kv_{Tj}}\right)^2 \right] \right\}. \quad (17)$$

We denote the real part of ω by ω_r and imaginary part by ω_i. Then, for instance, for waves with $\omega_r \sim \Omega_{He}$, the damping is mainly due to He ions and is given as

$$\frac{\omega_i}{\omega_r} \simeq \sqrt{\frac{\pi}{2}} \frac{\omega_{\mathrm{pHe}}^4}{k^4 c^4} \frac{\Omega_{\mathrm{He}}}{k v_{\mathrm{THe}}} \frac{\omega_r^2}{(\omega_r \Omega_{\mathrm{He}} - 2\Omega_{\mathrm{He}}^2)} \exp\left[-\frac{1}{2} \left(\frac{\omega_r - \Omega_{\mathrm{He}}}{k v_{\mathrm{THe}}} \right)^2 \right]. \tag{18}$$

For a H-He plasma with $n_{\mathrm{He}}/n_{\mathrm{H}} = 0.1$, $n_e = 10^8$ cm^{-3}, $T = 100$ eV, and $B = 100$ G, which are typical parameters for coronal magnetic tubes, the above equation gives $\omega_i/\omega_r \sim 10^{-370}$ for $\omega_r = 0.9\ \Omega_{\mathrm{He}}$. On the other hand, the growth rate of the modulational instability for the same wave with an amplitude $B_\perp/B_0 = 0.1$ is $\Gamma/\omega_r \simeq 0.03$. The damping rate is thus much smaller than the growth rates of the modulational instability.

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