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Soliton Formation in Spiral Galaxies

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Abstract

A galactic disk, consisting of collisionless stars, can be treated as a plasma under the influence of the collective gravitational force. In the present work, a comparison between the dynamics of galaxies and conventional plasmas has been carried out. We invoked the analogy between Kelvin-Helmholtz instability and the instability of a differentially rotating fluid due to two-dimensional perturbations, as well as the one between Rayleigh-Taylor instability in stratified fluids and the instability determined by Rayleigh's criterion for rotating flows. By linear analysis, we have a classification of the parameter space. In the regime of stable density waves, we have studied the nonlinear evolution of the density perturbation in the spiral structure of galaxies applying reductive perturbation method to the fluid dynamical description of galaxies. We compare two different forms of soliton-like waves described either by KdV or NLS equations.

Keywords:

galactic disk, instabilities, nonlinear density wave, soliton formation

1. Introduction

The mechanism of formation of the spiral patterns observed in most galaxies has not yet been fully understood. Since a galaxy is a star-gas mixture sup-ported by gravitation, pressure, rotation and magnetic field, collective phenomena and structures created there are rather rich and complex [1]. The aim of this paper is to highlight analogies among galaxies, plasmas and neutral fluids in linear and nonlinear descriptions of waves excited in ambient rotations (or shear flows). In Section **2**, various instabilities that occur in different model of galaxies are discussed by comparing some of them with similar phenomena occurring in stratified fluids or plasmas. In order to study nonlinear dynamics of galaxies, we start from the linear theory, and summarize the analogy between stability criteria for water-waves dynamics and dynamics of galaxies. Some results of linear stability analysis will be used to define the parameter regime for the nonlinear model discussed in the latest part of this paper. In Section **3**, one-dimensional motion in the presence of rotation and pressure is considered in the nonlinear regime, and a KdV-type soliton solution is derived. This solution is compared with that of a pressureless, and rotationless model [2,3].

2. Stability Analysis

2.1 Stability Criterion for Galaxies

There are some different casuals of instabilities in galactic systems. Jean's criterion that gives the necessary and sufficient condition for the stability of a system of gravity and pressure (no rotation) [4] is

$$k^2c^2 > 4\pi G\rho \,, \qquad (1)$$

where k is the total wave number, c is the sound velocity, G is gravitational constant and ρ is the mass density.

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For a uniformly rotating infinite-length cylindrical column, Chandrasekhar proved that Jean's criterion is unaffected by rotation, except for modes with wave-numbers perpendicular to the axis of rotation [5]. For these modes Chandrasekhar's criterion is

$$k^2c^2 > 4\pi G\rho - 4\Omega^2 , \qquad (2)$$

where Ω denotes angular velocity of rotation. For waves propagating in the direction which is perpendicular to the rotation axis, there is no gravitational instabilities if $\Omega^2 > \pi G\rho$. In any other direction of the wave vector, gravitational instability occurs if Jean's criterion is satisfied. Finite thickness of the disk is responsible for the appearance of other critical wavelength since gravitational potential in that case is proportional to $(4\pi G\rho)/(k^2 + T^{-2})$ [1], where T is the thickness. In the limit of $T \to \infty$, Chandrasekhar's criterion is restored. For differential rotation, linearizing equations and assuming plane-wave type variation as $f_1 = f_1(r)e^{i(kr+m\varphi-\omega t)}$, we obtain the dispersion relation [6]

$$(\omega - m\Omega)^2 = \kappa^2 + k^2c^2 - 2\pi G\rho k , \qquad (3)$$

where $\omega - m\Omega$ is the Doppler shifted frequency and κ is epicyclic frequency due to differential rotation $\kappa^2 = 2\Omega(2\Omega + r(\mathrm{d}\Omega/\mathrm{d}r))$. For a pressureless medium, $c = 0$, $(\omega - m\Omega)^2$, become negative if $\kappa^2 < 0$, so the disk is unstable. This is the rotational instability due to exponentially growing departure of particles from circular orbits, and the growing rate is given by κ. The criterion for the rotation instability is

$$\frac{\mathrm{d}\Omega}{\mathrm{d}r} < 0 \text{ or } \left|\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right| < \frac{2\Omega}{r}$$

and finally

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^2\Omega)^2 < 0 . \qquad (4)$$

This criterion corresponds to Rayleigh's criterion for instability for rotational flow, and has analogy with stratified fluid at rest or with constant basic flow.

2.2 Stability Criterion for Rotational Flow

Governing equations for rotational flow are

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + 2(\boldsymbol{\Omega}\times\boldsymbol{r}) - \Omega^2\boldsymbol{r} = -\nabla p,$$
$$\nabla\cdot\boldsymbol{v} = 0 . \qquad (5)$$

After linearization and applying normal mode analysis assuming that all perturbations has the form as $f' = f'(r)e^{i(m\varphi+kz)-\omega t}$. $v_{r0} = 0$, $v_{\varphi 0} = V = \Omega(r)/r$, $v_{z0} = 0$ describes equilibrium. Equation for stability appears in following form

$$(\omega - im\Omega)^2 \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{m^2 + k^2r^2}\left(\frac{\mathrm{d}v_r'}{\mathrm{d}r} + \frac{v_r'}{r}\right)\right)$$
$$-\left((\omega - im\Omega)^2 + \frac{k^2r^2\Phi(r)}{m^2 + k^2r^2}\right.$$
$$\left. + im(\omega - im\Omega)r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}V/\mathrm{d}r + V/r}{m^2 + k^2r^2}\right)\right)v_r' = 0 \qquad (6)$$

together with boundary condition $v_r' = 0$, $r = r_1$, $r = r_2$. $\mathrm{d}V/\mathrm{d}r + V/r$ is basic vorticity and $\Phi(r) = \frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\,\mathrm{d}/\mathrm{d}r(r^2\Omega)^2$ is Rayleigh's discriminant [7] and corresponds to κ^2 in galaxy's dispersion relation, $\Phi(r) = \kappa^2/r^3$. Rayleigh's criterion is the necessary and sufficient condition for stability for axisymmetric perturbations, but is invalid for non-axisymmetric ones. Setting $\partial/\partial\varphi = 0$ and $m = 0$, we obtain stability equation

$$\frac{\mathrm{d}^2 v_r'}{\mathrm{d}r^2} + \frac{\mathrm{d}\Omega}{\mathrm{d}r}\left(\frac{v_r'}{r}\right) - k^2 v_r' - \frac{k^2}{\omega^2}\Phi(r)\,v_r' = 0 . \qquad (7)$$

Eigenvalues k^2/ω^2 are all negative if $\Phi(r) > 0$ allover the given interval $r_1 < r < r_2$. On the other hand, treating the two-dimensional perturbations, assuming that $\partial/\partial z = 0$ or $k = 0$, stability equation becomes

$$\omega\left(\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r}\right) - \frac{m^2}{r^2}\right)v_r'$$
$$-\frac{im}{r}\left(\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\Omega}{\mathrm{d}r} + 2\Omega\right)\right)v_r' = 0 , \qquad (8)$$

with boundary condition $v_r' = 0$, $r = r_1$, $r = r_2$. Now, we can speak only about necessary condition for instability,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\Omega}{\mathrm{d}r} + 2\Omega\right) \le 0 , \qquad (9)$$

which means that gradient of basic vorticity has to change sign at least once in given domain. This result is closely related with Kelvin-Helmoltz instability. In order to compare last result with Kelvin-Helmoltz instability, we introduce the stream function ψ for two-dimensional perturbations and search for solutions in the form $\psi'(r,\varphi,t) = \psi(r)e^{\omega t + im\varphi}$, where the amplitudes of the perturbed velocities are $v_r'(r) = im/r\ \psi(r)$, $v_\varphi'(r) = -\psi'(r)$. Stability equation is the same as (8). Multiplying this equation by complex conjugate ψ^* and integrating in the given domain, we obtain

$$\omega_r m \int_{r_1}^{r_2} \frac{r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r} + 2\Omega\right)}{\left|\omega + im\Omega\right|^2}\left|\psi\right|^2 \mathrm{d}r = 0 , \qquad (10)$$

which gives the necessary condition for instability.

2.3 Stability Criterion for Parallel Flow

Governing equations for parallel flow are the same as for rotational flow, but the basic state is given as $\boldsymbol{v} = U(z)\boldsymbol{e}_x$, $\rho = const$, $p = p_0$. Introducing stream function ψ again, for two-dimensional perturbations, we search for solution in the form $\psi'(x,z,t) = \psi(z)e^{i(kx-\omega t)}$. The imaginary part of the stability equation gives the necessary condition for instability as

$$\frac{\omega_i}{k}\int_{z_1}^{z_2}\frac{U''}{\left|U-\frac{\omega}{k}\right|^2}|\psi|^2\,dz = 0\,. \tag{11}$$

For Kelvin-Helmholtz instability, $U(z) \neq 0$, Eq. (11) shows that background flow has to change sign somewhere in the given domain, which corresponds to condition for two-dimensional perturbations in rotational flow. For stratified fluid with basic state defined as $U(z) = const$, $\rho = \bar{\rho}(z)$, $p = p_0 - g\int_{z_1}^{z_2}\bar{\rho}(z')dz'$, where g is the acceleration of gravity, Rayleigh's stability equation is

$$\left(\frac{\omega}{k}\right)^2\left(\frac{d^2}{dz^2}-k^2\right)\psi + N^2\psi = 0 \tag{12}$$

with boundary condition $k\psi = 0$, $z = z_1$, $z = z_2$. Instability in this case is called Rayleigh-Taylor instability. Rayleigh proved that necessary and sufficient condition for stability is $N^2 > 0$ everywhere in the given domain. This condition corresponds to the Rayleigh's criterion for axisymmetric perturbations in rotation flow, and to the condition in the case of the galaxies, so $N^2 = -g\,\frac{d\bar{\rho}/dz}{\bar{\rho}}$ corresponds to the Rayleigh's discriminant $\Phi(r)$, and to the epicyclic frequency κ^2. Physically, it means that for rotational flow (as well as for galaxies), whenever the z component of angular momentum decrease outward, instability occurs. This property is analogous to the instability that appears in stratified fluid whenever lighter fluid is locally below heavier fluid.

3. Nonlinear Analysis for One-dimensional Motion with Pressure and Rotation

3.1 Fluid Description of Galaxy

In Section **2**, we have summarized dispersion relations of linearized fluid models for various sources of instabilities. In this section, we study nonlinear waves in a gravitational rotating system, with two dimensional cylindrical geometry (corresponding to Sec. **2.1**). We consider a fluid in which fluid elements interact only through the self-gravity and pressure. In the cylindrical coordinates, the governing equations are written as

$$\begin{aligned}
&\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r)+\frac{1}{r}\frac{\partial}{\partial\varphi}(\rho v_\varphi)=0\\
&\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_r}{r}\frac{\partial v_\varphi}{\partial\varphi}-\frac{v_\varphi^2}{r}=-\frac{\partial\phi}{\partial r}-\frac{\partial}{\partial r}\left(\frac{p}{\rho}\right)\\
&\frac{\partial v_{\varphi r}}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\varphi}{r}\frac{\partial v_\varphi}{\partial\varphi}+\frac{v_r v_\varphi}{r}\\
&\qquad=-\frac{1}{r}\frac{\partial\phi}{\partial r}-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{p}{\rho}\right)\\
&\frac{1}{r}\frac{\partial\phi}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2}=4\pi G\rho\\
&p=K\rho^\gamma
\end{aligned} \tag{13}$$

Here we assume $v_r \ll v_\varphi$ and $\Omega = const$. We consider a density wave that propagates in the φ direction and approximate spatial derivative as $\frac{1}{r}\frac{\partial}{\partial\varphi}=\frac{\partial}{\partial x}$. Then the previous set of equations reduces to

$$\begin{aligned}
&\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)=0\\
&\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-K\gamma\rho^{\gamma-2}\frac{\partial\rho}{\partial x}-\frac{\partial\phi}{\partial x}\\
&\frac{\partial^2\phi}{\partial x^2}=\rho-v^2
\end{aligned} \tag{14}$$

where v is the x component of the velocity and all variables are normalized as: $\rho = \rho_0\bar{\rho}$, $p = 2\pi G\rho_0^2R^2\bar{p}$, $v = (2\pi G\rho_0)^{1/2}R\bar{v}$, $\phi = 2\pi G\rho_0R^2\bar{\phi}$, $x = R/\sqrt{2x}$, $t \doteq (2\pi G\rho_0)^{-1/2}\bar{t}$. We have supposed politropic fluid and that the variations of ρ and p take place adiabatically:

$$\begin{aligned}
&\frac{1}{\rho}\nabla p=\nabla\left(\frac{\gamma K}{\gamma-1}\rho^{\gamma-1}\right)\ \text{ for }\ \gamma\neq 1\ \text{ or}\\
&\frac{1}{\rho}\nabla p=\nabla\left(\frac{\gamma K}{\gamma-1}\rho^{\gamma-1}\right)\ \text{ for }\ \gamma\neq 1\,,\\
&\frac{1}{\rho}\nabla p=\nabla(K\log\rho)\ \text{ for }\ \gamma=1\,.
\end{aligned} \tag{15}$$

Linearizing the above system and assuming that all quantities are proportional to $e^{i(\omega t-kx)}$, we obtain the dispersion relation as

$$(\omega-k)^2k^2+k^2-\gamma k^4-2(\omega-k)k=0\,. \tag{16}$$

In the limit $k \gg \omega$ Eq. (16) coincides with the dispersion relation obtained in [2], for a rotationless case:

$$\omega'^2 = K\gamma k^2 - 1\,,$$

where

$$\omega' = \omega - k\,,\quad K = 1\,. \tag{17}$$

Waves with $k < (K\gamma)^{-1/2}$ have an imaginary frequency which means the exponential growth. Solving the dispersion relation (16) with respect to ω' and differentiating it with respect to k, we obtain group velocity V:

$$V=\frac{d\omega}{dk}=-\frac{1}{k^2}(1\pm\sqrt{1-k^2+K\gamma k^4})+\frac{1}{k}\frac{1}{\pm 2\sqrt{1-k^2+K\gamma k^4}}(-2k+4K\gamma k^3)+1 \quad (18)$$

Taking the limit $k \rightarrow 0$, which means that we consider $k \ll \omega$, only the branch with negative sign make sense, so $V = 3/2$, which corresponds to the same problem but pressureless.

3.2 One-dimensional Soliton

In the previous Sec. **3.1**, we have modified Chandrasekar criterion (discussed in Sec. **2.1**, Eq. (2)) for the waves propagating in the perpendicular direction. We consider only the waves that are linearly stable, i.e. $0 < k < (K\gamma)^{-1/2}$, and apply the reductive perturbation method [8] as:

$$\xi=\varepsilon^{\frac{1}{2}}(x-V_t), \quad \tau=\varepsilon^{\frac{3}{2}}t, \quad (19)$$

$$\rho=1+\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\varepsilon^n\rho^{(n,m)}(\xi,\tau)E^m$$
$$v=1+\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\varepsilon^n v^{(n,m)}(\xi,\tau)E^m$$
$$\phi=\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\varepsilon^n\phi^{(n,m)}(\xi,\tau)E^m \quad (20)$$

where ε is a small parameter, and $E = e^{i(\omega t-kx)}$ with k belonging to the linearly stable domain.

From the lowest order set of equations is obtained:

$$v^{(1,0)}=\frac{1}{2}\rho^{(1,0)}, \quad \phi^{(1,0)}=\frac{1-4K\gamma}{4}\rho^{(1,0)}, \quad V=\frac{3}{2}. \quad (21)$$

From the order of $\varepsilon^{5/2}$, a KdV type equation is obtained as following:

$$\frac{\partial}{\partial\tau}\phi^{(1,0)}+\frac{3}{1-4K\gamma}\phi^{(1,0)}\frac{\partial}{\partial\xi}\phi^{(1,0)}-\frac{1-4K\gamma}{8}\frac{\partial^3}{\partial\xi^3}\phi^{(1,0)}=0 \quad (22)$$

This type of nonlinear equation has a solution in the form [9]

$$\phi(\xi,\tau)=\phi_\infty+a\,\mathrm{sech}^2[(\xi-V\tau)(\frac{a}{12b})^{\frac{1}{2}}], \quad (23)$$

where ϕ_∞ denotes the boundary value of $\phi^{(1,0)}$ at $(\xi - V\tau) \rightarrow \pm\infty$, a is amplitude of the wave relative to the constant solution ϕ_∞ at infinity, and $b = 27/8(1 - 4K\gamma)^2$.

It is clear that the KdV equation is obtained where there is no carrier wave, i.e. $m = 0$. On the other hand, a solution for rotationless case appears as a solution of NLS equation [2,3], where carrier wave is included. The first situation corresponds to a solution for the ion-acoustic wave in the plasma theory. Poisson's equation in this case, has the form $\frac{\partial^2\phi}{\partial x^2}= n_i - n_e = n_i - e^{\phi}$, where the last term in the right-hand side is not a constant. However, the second situation corresponds to the Langmiur wave, because Poisson's equation is $\frac{\partial^2\phi}{\partial x^2}= n_e - n_i$, and last term, n_i is a constant. Main difference in analogy between the ion-acoustic wave and the wave for this one-dimensional model for galaxies, with pressure and uniform rotation, is the form of non-constant last term in Poisson's equation, and therefore its influence in appearance in the equation of motion. Note that in galaxy case this term is v^2 and its origin comes from the uniform rotation. This term produces essential coupling between higher order harmonics and introducing of carrier wave can cause unnecessary coupling of harmonics in variables expansion.

On the other hand, for galaxy model without rotational effects, Poisson's equation has the form similar to the Langmiur wave, with the constant last term. In this case, introducing the carrier wave is necessary to produce coupling, since there is no term that is involved in the equation of motion.

4. Concluding Remarks

We have investigated the linear stability of different models of galaxies and compared different types of instabilities that can occur in such models. Some analogues are found between galaxy-type instability and instabilities that are well established in the fluid theory such are Rayleigh's criterion for rotational flow, and Rayleigh-Taylor instability for stratified fluid at rest or with constant basic flow. In Section **3**, we generalized the well-known Chandrasekar criterion for the linear waves propagating in the direction perpendicular to the Ω, and obtained a KdV equation that gives soliton solutions.

Comparison with plasma waves, and also with NLS type of solution for rotationless model has been made.

In the last one, NLS equation [2] changes type from focusing to defocusing through the critical wavenumber $k_c = \sqrt{3(\gamma + 2)/4\gamma(\gamma + 1)} > (\gamma)^{-1/2}$. This range is out of our consideration, since it is a linearly unstable wavenumber. Here, carrier wave is responsible for essential coupling. However, for the study of galactic structures, thin-disk geometry plays an essential role (Sec. **2.1**, Eq. (3) and Sec. **2.2**, Eq. (9)) through new approximation for Poisson's equation. An NLS type equation can be derived [10] for linearly stable waves satisfying the dispersion relation (3), which has dark soliton solutions [11]. We will discuss elsewhere other type of nonlinear waves in a thin-disk geometry.

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