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A Walk in the Parameter Space of L–H Transitions without Stepping on or through the Cracks

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Abstract

A mathematically and physically sound three-degree-of-freedom dynamical model that emulates low- to high-confinement mode (L–H) transitions is proposed on the basis of a singularity theory critique of earlier fragile models. It is found to contain two codimension 2 organizing centers and two Hopf bifurcations, which underlie dynamical behavior that has been observed but not mirrored in previous models.

Keywords:
L–H transitions, dynamical model, singularity, stability

1. Introduction

A unified, low-dimensional description of the dynamics of L–H transitions would be a valuable aid for the control of confinement in fusion plasmas. In this work we report significant progress toward this goal by developing the singularity theory approach this problem that was introduced in [1].

The title of this paper refers to the philosophy of singularity theory [2] as applied to dynamical models: that parameter space should be smooth and continuous, and parameters should be independent and not fewer than the codimension of the system. (See Refs. [2] and [3] for explanations of the mathematical terminologies and operations in this paper.)

Since 1988 [4] many efforts have been made to derive unified low-dimensional dynamical models that emulate L–H transitions and/or associated oscillatory behavior [5-21]. However, as was shown in [1], the models often founder at singularities. Consequently, much of the discussion in the literature concerning their bifurcation properties is qualitatively inadequate. For example, the model in Ref. [13] is claimed to reflect a second-order phase transition, but it is easily shown [1] that a perturbation of arbitrary size dissolves the putative transition point. A similar problem also besets the DLCT (Diamond-Liang-Carerras-Terry) model [8], which we examine in this paper. The notion that the L–H transition is a second-order phase transition has entered the literature [22,23] simply because these and similar models neglected perturbing terms that correspond to some essential physics, such as fulfilling a symmetry-breaking imperative. We say that such models are fragile, and cannot predict the behaviour of the system they purport to represent, because a perturbation changes profoundly the character of the solutions.

Our analysis of the bifurcation structure of the DLCT model finds that it needs two major operations to give it mathematical consistency: (1) a degenerate singularity is identified and unfolded, (2) the dynamical state space is expanded to three dimensions. We then analyse the enhanced model obtained from these operations, the BD (Ball-Dewar) model, and find it

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consistent with many known features of L–H transitions.

2. Bifurcation Structure of the DLCT Model

This paradigmatic 2-dimensional model [8] couples the turbulence and flow shear dynamics:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \gamma N - \alpha FN - \beta N^2 \quad (1)$$

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \alpha FN - \mu F. \quad (2)$$

N is the mean square level of density fluctuations, F is the square of the averaged $\boldsymbol{E} \times \boldsymbol{B}$ poloidal flow shear. The fluctuations grow at rate γN and are damped quadratically at rate βN^2. The exchange coefficient α is related to the Reynolds stress, and the damping rate μF is due to viscosity. (Minor changes to the original notation have been made.)

Following the procedure in [1] we set the time derivatives in eqs. 1 and 2 to zero, select the state variable and bifurcation parameter to form the bifurcation equation $g = X(F, \gamma) = (F\alpha(\gamma - F\alpha))/\beta - F\mu$, and identify the singular points where $g = g_F = 0$. We find the unique physical singularity $(F, \gamma)_T = (0, \beta\mu/\alpha)$, which satisfies the additional defining conditions for a transcritical bifurcation:

$$g_\gamma = 0, \ g_{FF} \neq 0, \ \det d^2 g < 0, \quad (3)$$

where d^2g is the Hessian matrix. The bifurcation diagram showing the transcritical point T is plotted in Fig. 1a. (In these diagrams stable solutions are plotted with continuous lines and unstable solutions with dashed lines.)

However, Fig. 1a is incomplete because of this generic property of the transcritical bifurcation: *it is non-persistent to perturbation.* To break the symmetry under $v' \to -v'$, where v' represents the poloidal shear flow, we introduce the perturbation term $\varphi F^{1/2}$. (Note that $F \propto |v'|^2$.) Thus, the modified DLCT model consists of eq. 1 and

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \alpha FN - \mu F + \varphi F^{1/2}. \quad (4)$$

The perturbation term represents an inevitable source of shear flow. It can arise from non-ambipolar ion orbit losses that produce a driving torque, which can be quite large [24]. In fact some early models relied exclusively on a nonlinear ion orbit loss rate [4,5]. However, this alone cannot explain turbulence suppression. Here we treat the term as part of a more complete dynamical picture and assign it a simple form in the small.

Bifurcation diagrams for increasing values of φ are plotted in Fig. 1b–d. We see that solution of one problem causes another: the perturbation does indeed unfold T, but it releases *another* degenerate singularity T^s. Before proceeding with a treatment of T^s we highlight three important issues:

1. Since φ is inevitably nonzero in experiments, no distinct transition occurs near T, second-order or otherwise, contrary to what is stated in [8].
2. Both N and F change continuously with γ in the same direction. Figure 2, where N is the state variable, should be compared with Fig. 1c. The model therefore does not emulate turbulence stabilization by the shear flow, contrary to what is stated in [8].
3. To ascertain whether the model can exhibit periodic dynamics as stated in [8] we look for a pair of purely

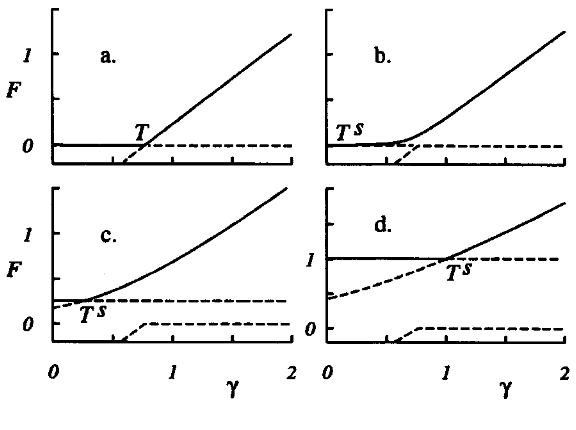

Fig. 1 The perturbation φ unfolds T but introduces T^s. $\alpha = 1$, $\beta = 0.77$, $\mu = 1$. a. $\varphi = 0$, b. $\varphi = 0.05$, c. $\varphi = 0.5$, d. $\varphi = 1$. The region $F < 0$ is unphysical but is included to make the nature of T and T^s clearer.

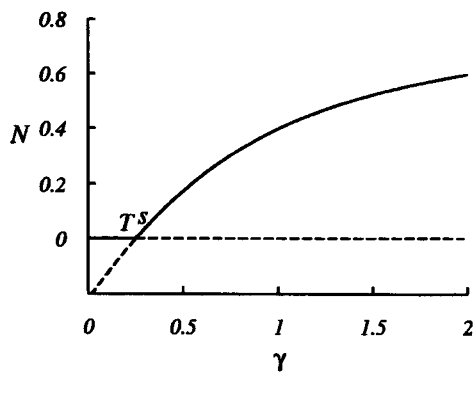

Fig. 2 Same as Fig. 1(c) except with N as the state variable. The region $N < 0$ is unphysical but is included to make the nature of T^s clearer.

complex conjugate eigenvalues. Applying the defining conditions for Hopf bifurcations,

$$g = \mathrm{tr}J = 0,\ \det J > 0,\ \frac{\mathrm{d}}{\mathrm{d}\gamma}\mathrm{tr}J \neq 0,$$

where J is the Jacobian matrix, to eqs. 1 and 4 we find that $\det J < 0$ where the equalities are fulfilled, therefore limit cycles arising from Hopf bifurcations cannot occur. Although periodic behavior arising from rare and pathological causes is still possible (if we apply Dulac's criterion [3] to eqs. 1 and 4), we have not found oscillatory solutions numerically.

Returning to the new singularity T^s we find that it is also a transcritical point according to eq. 3. Does the DLCT model therefore require a *second* perturbation, to eq. 1, to unfold T^s? There is no matching physics to justify this. Another possibility is that T^s is spurious, created by an unwarranted collapse of a larger state space. This idea leads to a suggestion that *is* supported by the physics, that another variable is intrinsic to a low-dimensional description of L–H transition dynamics.

3. Intrinsic 3-Dimensional Dynamics of L–H Transitions

We introduce the the third variable by assuming $\gamma = \gamma(P)$, where P is the pressure gradient, as have several other authors [9,12-14,20]. Assuming a simple evolution of P and $\gamma(P) = \gamma P$, we obtain an augmented model, comprised of eq. 4 and

$$\varepsilon\frac{\mathrm{d}P}{\mathrm{d}t} = q - \gamma PN \tag{5}$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \gamma PN - \alpha FN - \beta N^2. \tag{6}$$

In eq. 5 q is the power input and ε is a dimensionless regulatory parameter. For $\varepsilon \ll 1$ or $\varepsilon \gg 1$ the system can evolve in two timescales:

1. For $\varepsilon \to 0$, $\varepsilon dP/dt \approx 0$ and $P \approx q/(\gamma N)$. The system collapses smoothly to eq. 4 and

$$\frac{\mathrm{d}N}{\mathrm{d}t} = q - \alpha FN - \beta FN^2. \tag{7}$$

The organizing center is T, the spurious T^s is non-existent, and there are no Hopf bifurcations. For $\varepsilon \gg 1$ we define $\delta \equiv 1/\varepsilon$, and taking the limit as $\delta \to 0$ we recover the same form as eq. 1,

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \gamma P_0 N - \alpha FN - \beta N^2, \tag{1'}$$

as well as eq. 4, along with the "good" bifurcation T and the "bad" bifurcation T^s, therefore we suggest this is a non-physical limit for ε.

2. In "fast" time $\tau \equiv t/\varepsilon$ and, recasting the system accordingly, it can be seen that the dynamics becomes 1-dimensional in P in both limits.

The organizing center of the complete problem, eqs. 4, 5, and 6, is the unique transcritical bifurcation $(F,q,\varphi)_T = (0, \beta\mu^2/\alpha^2, 0)$ and T^s is non-existent. We now have the bones of an improved model, but it still does not emulate the following characteristics: (a) hysteresis, since there is no non-trivial point where $g_{FF} = 0$, (b) oscillations in H-mode. Evidently we need more nonlinearity to produce enough competitive interaction.

A likely stimulus to nonlinear behavior is the viscosity, which in [12] was considered to be the sum of neoclassical and anomalous or turbulent contributions, both dependent on the pressure gradient. We shall adopt this bipartite form and in eq. 4 take

$$\mu = \mu(P) = \mu_{\mathrm{neo}}P^n + \mu_{\mathrm{an}}P^m. \tag{8}$$

Equations 5, 6, 4, and 8 comprise the BD model. The value of –3/2 is used for the exponent n, as in [12], which arises from the temperature dependence of the ion viscosity in a low collisional régime derived from neoclassical theory [25]. In [26] the anomalous viscosity is given with a $P^{3/2}$ dependence, but is also influenced by a P-dependent curvature factor. We have taken m = 5/2 to simulate these effects. The fidelity of the qualitative structure of the system to variations in m and n is discussed briefly in Sec. **5**.

4. Bifurcation Structure of the BD Model

The bifurcation problem may be expressed in terms of F as $g = \alpha FN - \mu F + \varphi F^{1/2}$, with N and P given by the zeros of the RHS of eqs. 6 and 5. It contains two codimension 2 organizing centers:

1. The defining conditions for the pitchfork, $\wp$,

$$g = g_F = g_{FF} = g_q = 0,\ g_{FFF} \neq 0,\ g_{Fq} \neq 0,$$

give $(F, q, \beta, \varphi)_{\wp} = (0, 8\mu_{\mathrm{an}}^{1/8}\,\mu_{\mathrm{neo}}^{7/8}\gamma/(7^{7/8}\alpha), (7^{3/8}\alpha\gamma)/(8\mu_{\mathrm{an}}^{5/8}\,\mu_{\mathrm{neo}}^{3/8}), 0)$. Away from the critical value of β, $\wp$ becomes a transcritical T^l.

2. Another transcritical point T^u occurs at $(F, q, \varphi)_{T^u} = (0, P^2\gamma/\beta, 0)$. T^l and T^u are annihilated at a second codimension 2 singularity, $\mathcal{A}$, at which the defining conditions

$$g = g_F = g_q = \det d^2g = 0,\ g_{FF} \neq 0,\ g_{Fq} \neq 0$$

give $(F, q, \beta, \varphi)_{\mathcal{A}} = (0, 8(7^{1/8})\mu_{\mathrm{neo}}\gamma/(3\alpha(\mu_{\mathrm{neo}}/\mu_{\mathrm{an}})^{1/8}), 3(\mu_{\mathrm{neo}}/\mu_{\mathrm{an}})^{5/8}\alpha\gamma/(8(7^{5/8}\mu_{\mathrm{neo}})), 0)$.

The partially perturbed and fully perturbed bifurcation diagrams are plotted in Figs 3a. and b. respectively. There are also *two* Hopf bifurcations linked by a branch of stable limit cycles, the amplitude trace of which is marked by dotted lines. This reflects the passage through an oscillatory regime with increasing power that has been observed in experiments as a feature of type III edge-localised modes [27,28]. However, the quantitative features of type III ELMs, such as the frequency spectrum, are not reproduced by this simple model.

Figure 4 illustrates one of the possible effects of poor turbulence dissipation (i.e. low β). It shows a direct transition to an oscillatory state, which has been observed in some experiments [29,30].

Figure 5, to be compared with Fig. 3b, shows that the BD model does indeed reflect shear flow suppression of turbulence. The turbulence decreases dramatically where the jump in the shear flow occurs, then begins to rise again as increasing power is input to the system. Near the unfolded T_u, at q ~ 170, the rise in turbulence becomes smaller as the turbulent viscosity damping (second term in eq. 8) takes over.

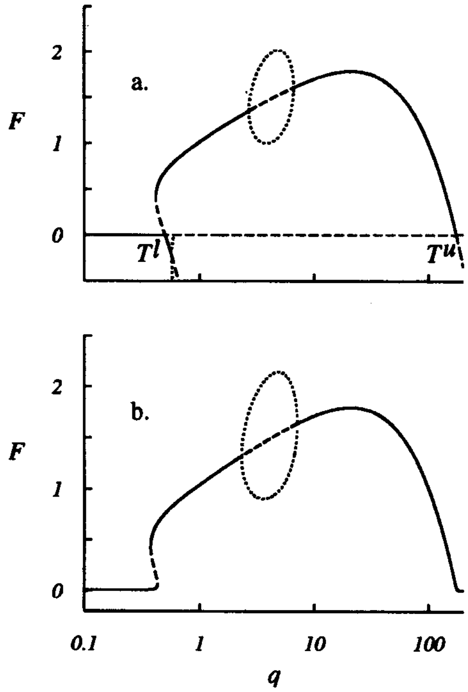

Fig. 3 Bifurcation diagrams of the BD model. $\alpha = 2.4$, $\beta = 1$, $\gamma = 1$, $\varepsilon = 1$, $\mu_{neo} = 1$, $\mu_{an} = 0.05$, $n = -1.5$, $m = 2.5$. a. Bifurcation structure of the partially perturbed system, with $\varphi = 0$. b. $\varphi = 0.05$. The unphysical branches are not plotted in b.

5. Discussion and Conclusions

A dynamical model that emulates much of the typical behavior around L–H transitions has been elicited from an earlier flawed model, by considering the relationship between bifurcation structure and the physics of the process. Built in to this model are the following attributes of L–H transitions:

1. Discontinuous, hysteretic transitions.
2. The onset *and* abatement of oscillatory behavior, via the two Hopf bifurcations in H-mode, and a transition directly into oscillatory H-mode.
3. Turbulence suppression by the shear flow.
4. Turbulence generation via non-ambipolar losses.

A maximum in the shear flow followed by a decrease as the power input increases is also predicted.

In this model it is not necessary to assume any dependence of the shear flow/turbulence exchange or

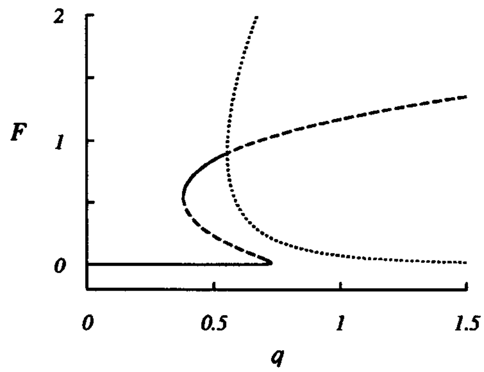

Fig. 4 Under extreme conditions, when the turbulence dissipation rate is relatively low, the jump at the lower limit point can occur directly to an oscillatory state on the H-mode branch. $\beta = 0.1$, other parameters as for Fig. 3.

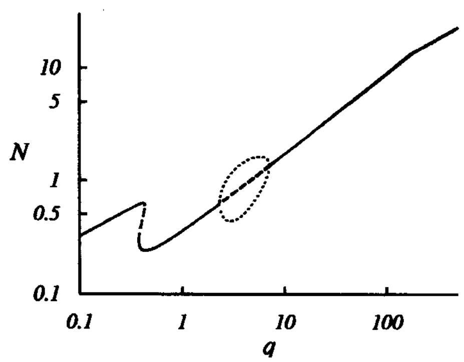

Fig. 5 Same as Fig. 3b, except with N as state variable.

turbulence dissipation rates on the pressure gradient, as in ref. [12], to obtain the qualitative structure and dynamics described here. If we write $\alpha = \alpha(P) = \alpha P^r$ and $\beta = \beta(P) = \beta P^s$ we find that the structural properties of the model are insensitive to values of r and s between ~ -1 and ~ 1, hence we have simply chosen $r,s = 0$. We also find that the qualitative structure is unchanged for any $m \geq 0$ in the viscosity function of eq. 8, when $n < -1$, or for any $n < -1$, when $m \geq 0$. These results will be presented in more detail elsewhere.

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