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Forced Magnetic Reconnection due to Boundary Perturbation

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Abstract

Boundary layer analysis of forced magnetic reconnection due to a boundary perturbation is revised. This revised analysis introduces correct asymptotic matching to take into account the effect of inertia in the inner layer precisely. The initial evolution of a new reconnection process is characterized by some significant features. One is that a reconnected flux increases on the same time scale as the boundary perturbation, which excludes the Sweet-Parker time scale obtained by use of the constant-ψ asymptotic matching. Another is that an induced surface current on a resonant surface is in such a direction as to oppose the progress of the reconnection. Moreover an equation for the time evolution of the reconnected flux is proposed in terms of an integral equation.

Keywords:

MHD, boundary layer theory, correct asymptotic matching, magnetic reconnection

1. Introduction

In tokamak plasmas, the boundary perturbation is caused by resonant magnetic field errors. The error field is the small deviation from axial symmetry of the magnetic field lines and it perturbs the plasma boundary to form magnetic islands [1]. The boundary perturbation also models the presence of a magnetic signal produced from another magnetohydrodynamic (MHD) event, such as a sawtooth crash. This boundary perturbation can produce the seed islands for the neo-classical tearing mode [2]. For example, the magnetic signal with poloidal and toroidal numbers $m = 3$, $n = 2$ is produced by an $m = n = 2$ magnetic signal associated with a sawtooth crash in combination with toroidal geometry. Even if a magnetic equilibrium is stable for resistive modes, an externally imposed boundary perturbation gives rise to magnetic reconnection called forced reconnection [1].

The response of the plasma to the applied boundary perturbation is determined by a time dependent coefficient, called the reconnected flux. The time development of the reconnected flux is calculated by use of boundary layer theory as an initial value problem [1-6]. In the previous analysis, the time scale of the initial evolution of forced reconnection is believed to be the Sweet-Parker time scale. We reveal that this time scale stems from the particular matching conditions which are valid only in the constant-ψ approximation; the effect of the inertia in the inner layer is neglected in this matching condition. Therefore the results in previous works do not reflect the effect of the inertia correctly.

We introduce the appropriate asymptotic matching and adopt the exact solution for the inner layer equation to take into account precisely the effect of inertia in the inner layer [7]. The improved reconnection process

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exhibits a new time evolution of the magnetic islands and of the surface current induced on the resonant surface.

2. Model and Boundary Layer Theory

We consider the response of a slab of incompressible plasma to an applied boundary perturbation. The plasma is supposed to be bounded by two parallel perfectly conducting walls. The magnetic field is represented by $\boldsymbol{B} = B_T\boldsymbol{e}_z + \boldsymbol{e}_z \times \nabla\psi$, where B_T stands for a uniform toroidal field and ψ is a magnetic potential. We assume that the xy-plane is normal to the toroidal field, while the y-axis is parallel to the wall and the x-axis normal to it.

2.1 Outer region

The outer region is governed by the ideal MHD equilibrium equation,

$$\nabla \times (\boldsymbol{j} \times \boldsymbol{B}) = 0, \tag{1}$$

where $\boldsymbol{j} = \nabla \times \boldsymbol{B}/4\pi$ is the current density.

In the absence of the boundary perturbation there is a static equilibrium which is represented by an even function, $\psi = \psi_0(x)$, subjected to the boundary conditions $\psi_0(\pm a) = const.$, where $x = \pm a$ expresses the boundaries of the plasma. This equilibrium is assumed to have the resonant surface at the center of the plasma, $\psi_0(0) = 0$, and is supposed to be stable for the usual tearing mode.

Let us impose a boundary perturbation on the original static equilibrium. It is described by means of in terms of a deformed plasma boundary as

$$\psi(x = \pm(a - \delta \cos ky)) = const.,$$

where k and δ are the wave number and the amplitude of the boundary deformation, respectively. The boundary perturbation is assumed to be weak, $\delta << a$, such as the error fields in a tokamak. In this work, we consider a time varying boundary perturbation, $\delta = \delta(t/\tau_e)$, instead of a suddenly imposed boundary perturbation. The time scale of the deformation τ_e is assumed to be much slower than the Alfvén time scale but much faster than the resistive time scale, $\tau_A << \tau_e << \tau_R$; τ_A and τ_R are defined later.

The magnetic potential perturbed by the boundary deformation is written as

$$\psi(x,t) = \psi_0(x) + \psi_1(x,t)\cos ky, \tag{2}$$

where $\psi_1(x,t)$ denotes the perturbed part due to the boundary perturbation. Since the perturbation is imposed on a time scale much slower than the Alfvén time scale,the plasma is quasi-static and obeys the ideal MHD equilibrium equation except in the vicinity of the resonant surface, where $x = 0$. The ideal MHD equilibrium equation, Eq. (1), for the perturbation $\psi_1(x,t)$ is reduced to

$$\psi_0'(x)\left\{\frac{\partial^2 \psi_1(x,t)}{\partial^2 x} - k^2 \psi_1(x,t)\right\} - \psi_0'''(x)\,\psi_1(x,t) = 0, \tag{3}$$

with the boundary condition

$$\psi_1(\pm a, t) = \delta(t/\tau_e)\,\psi_0'(a) \equiv \psi_e(t/\tau_e).$$

The solution to this equation, $\psi_1(x,t)$, should be an even function of x, because Eq. (3) and the boundary condition are unchanged for $x \to -x$.

The solution to Eq. (3) is written as

$$\psi_1(x,t) = \psi_1(0,t)f(x) + \psi_e(t/\tau_e)g(x), \tag{4}$$

where $f(x)$ and $g(x)$ satisfy Eq. (3) respectively, and are subject to the boundary conditions $f(0) = 1, f(\pm a) = 0$ and $g(0) = 0$, $g(\pm a) = 1$ [2,5].

We consider the time evolution of the quasi-static equilibrium as an initial-value problem, by applying the Laplace transform to the outer-solution, Eq. (4). Then we derive the expansion to the outer solution as

$$\tilde{\psi}_1(x,s) = \tilde{\psi}_1(0,s) + \tilde{\psi}_1(x,s)\frac{\Delta'_{out}}{2}x + \cdots, \quad x \to +0, \tag{5}$$

where

$$\Delta'_{out}(s) = \frac{1}{\tilde{\psi}_1(0,s)}\left[\frac{d\tilde{\psi}_1(x,s)}{dx}\right]_{-0}^{+0} = \Delta'_0 + \Delta'_s\frac{\tilde{\psi}_e(s)}{\tilde{\psi}_1(0,s)}, \tag{6}$$

and where

$$\Delta'_0 = \left[\frac{df(x)}{dx}\right]_{-0}^{+0}, \qquad \Delta'_s = \left[\frac{dg(x)}{dx}\right]_{-0}^{+0}.$$

The first term, Δ'_0, is the tearing mode stability parameter in the absence of the boundary perturbation. %and for the ideal response to the boundary perturbation, respectively. Since the original static equilibrium is supposed to be stable, Δ'_0 is negative.

The quasi-static equilibrium, Eq. (4), is determined only by the reconnected flux $\psi_1(0,t)$, because $\psi_e(t/\tau_e)$ is assumed to be a given function. In order to obtain the reconnected flux we should investigate the dynamics in the vicinity of the resonant surface, i.e. the inner layer. In the analysis of the inner layer it is important to include not only the resistivity but also the inertia of the plasma correctly.

2.2 Inner layer

We apply the Laplace transform

$$\tilde{f}(x,s)=\int_0^{\infty} f(x,t)\, e^{-st}\mathrm{d}t, \tag{7}$$

to the linearized reduced MHD equations. The initial conditions for the perturbations are $\psi_1(x,0) = \varphi_1(x,0) = 0$, because there is no deformation of the boundary at $t = 0$, $\psi_e(0) = 0$. By stretching the variables in the vicinity of the resonant surface according to

$$\hat{x}=\frac{x}{\varepsilon a}, \quad \hat{s}=\tau_c s, \quad \varepsilon=\left(\frac{\tau_A}{\tau_R\, ka}\right)^{1/3}, \quad \tau_c=\frac{\tau_A}{\varepsilon ka},$$

the equations in the inner layer become

$$\hat{s}\,\frac{\mathrm{d}^2\tilde{\psi}_{in}}{\mathrm{d}\hat{x}^2}=-\hat{x}\,\frac{\mathrm{d}^2\tilde{\psi}_{in}}{\mathrm{d}\hat{x}^2}, \tag{8}$$

$$\hat{s}\,\tilde{\psi}_{in}-\hat{x}\,\tilde{\psi}_{in}=\frac{\mathrm{d}^2\tilde{\psi}_{in}}{\mathrm{d}\hat{x}^2}, \tag{9}$$

where

$$\tilde{\psi}_{in}(\hat{x},\hat{s})=\frac{\tilde{\psi}_1(x,s)}{\psi''_0(0)\,a^2}, \quad \tilde{\varphi}_{in}(\hat{x},\hat{s})=\frac{\tilde{\varphi}_1(x,s)}{v_A\, a},$$

$\varphi = \varphi_1(x,t)\sin ky$ is the stream function, $\tau_A = a/v_A$, $\tau_R = 4\pi a^2/\eta$ and $v_A = \psi''_0(0)a/\sqrt{4\pi\rho}$.

The inner-layer solution can be expanded asymptotically as

$$\tilde{\psi}_{in}(\hat{x},\hat{s})\approx\psi_\infty -\psi_\infty\frac{\pi\hat{s}^{5/4}}{8}\frac{\Gamma(\hat{s}^{3/2}/4-1/4)}{\Gamma(\hat{s}^{3/2}/4+5/4)}\hat{x}+\cdots, \tag{10}$$

for $\hat{x}\rightarrow+\infty$, where Γ is the Gamma function.

3. Reconnected Flux by Exact Asymptotic Matching

Demanding that the solution to the inner-layer equation matches asymptotically with the solution in the outer region yields the exact matching condition. which includes the effect of the inertia in the inner layer correctly. On the other hand the matching adopted in previous works is appropriate only in the constant-ψ approximation. In order to clarify this point, we rewrite the asymptotic expansion of $\tilde{\psi}_{in}$, Eq. (10), as

$$\tilde{\psi}_{in}(\hat{x})\approx\psi_\infty\left\{1+\frac{\Delta'_{in}}{2}x+\cdots\right\} \quad \hat{x}\rightarrow+\infty, \tag{11}$$

where

$$\Delta'_{in}(s)\equiv\frac{1}{\varepsilon a\,\psi_\infty}\left[\frac{\mathrm{d}\tilde{\psi}_{in}}{\mathrm{d}\hat{x}}\right]_{-\infty}^{\infty} =\frac{-\pi\hat{s}^{5/4}}{8\varepsilon a}\frac{\Gamma(\hat{s}^{3/2}/4-1/4)}{\Gamma(\hat{s}^{3/2}/4+5/4)}. \tag{12}$$

In the previous analysis [1-4], $[d\tilde{\psi}_{in}/d\hat{x}]_{-\infty}^{+\infty}$ is divided by $\tilde{\psi}_{in}(0,\hat{s})$ instead of ψ_∞ in Eq. (12). That is valid only in the constant-ψ approximation realized by neglecting the effect of plasma inertia in the inner layer, because the effect of inertia makes $\tilde{\psi}_{in}(0)$ deviate from ψ_∞. $\psi_\infty = -\chi_\infty$. Therefore the matching condition derived by the expansion, Eq. (11), reflects the effect of inertia in the inner layer correctly.

Asymptotic matching of Eqs. (5) and (11) yields the new Laplace-transformed reconnected flux

$$\tilde{\psi}_1(0,s)=\frac{\Delta'_s\,\tilde{\psi}_e(s)}{\Delta'_{in}(s)-\Delta'_0}. \tag{13}$$

4. Initial Evolution

In this section, the Laplace-transformed reconnected flux, Eq. (13) provides an initial evolution of the new forced reconnection process.

We consider a time dependent boundary perturbation as

$$\psi_e(t/\tau_e)=\frac{\psi''_0(0)}{2!}\frac{t^2}{\tau_e^2}+\cdots. \tag{14}$$

The Taylor-series expansion of the reconnected flux is given by Eq. (13) as

$$\psi_1(0,t)=-\frac{\Delta'_s}{\Delta'_0}\left\{\frac{\psi''_0(0)}{\tau_e^2}\frac{t^2}{2!}+\cdots\right\}, \tag{15}$$

where

$$\tau_a=\frac{-\Delta'_0}{\pi k}\,\tau_A, \quad \tau_c=\frac{\tau_A^{2/3}\,\tau_R^{1/3}}{(ka)^{2/3}}, \tag{16}$$

denote the ideal time scale and the typical time scale in the inner layer, respectively. This reconnected flux vanishes at $t = 0$ to the initial condition. We consider the time scale of the reconnection process. The initial evolution of the reconnected flux is dominated by the first term on the right hand side of Eq. (15), therefore its typical time scale is the same as the time scale of the boundary perturbation, τ_e, and does not include the Sweet-Parker time scale at all.

When the boundary perturbation is imposed, a surface current is induced at the resonant surface. The amount of surface current is represented by the total current in the inner layer and is equivalent to the finite jump of the y-component of the magnetic field at the resonant surface, $x = 0$,

$$\Delta B_y(t) \equiv \left[\frac{\partial \psi_1(x,t)}{\partial x} \right]_{-0}^{+0} \tag{17}$$

Substituting Eqs. (14) and (15) into Eq. (17) gives the initial evolution of the total current in the inner layer as

$$\Delta B_y(t) = -\Delta'_s \left\{ \frac{\psi''_e(0)}{3!} \frac{t^3}{\tau_a \tau_e^2} + \cdots \right\}. \tag{18}$$

In order to compare Eq. (18) with the result found in previous works [1,3,4], we consider the time evolution of the stability parameter $\Delta'(t) \equiv \Delta B_y / \psi_1(0,t)$. The stability parameter $\Delta'(t)$ is negative in the initial evolution and $\Delta'(t) = 0$ at $t = 0$, while it was claimed that $\Delta'(t) \to \infty$ with the positive sign at $t = 0$ in previous works. The sign of $\Delta'(t)$ determines whether the surface current stabilizes or destabilizes the magnetic islands. The negative sign of $\Delta'(t)$ or $\Delta B_y(t)$ implies that the surface current is induced so as to oppose the growth of the magnetic islands. However the push caused by the imposed boundary deformation overcomes this resistance to form the magnetic islands.

5. Evolution Equation for Reconnected Flux

In the preceding section we have obtained the initial evolution. In this section we propose a new method to determine the time evolution of the reconnected flux. By use of this method, we can obtain the time development subsequent to the initial evolution.

The inversion of the Laplace transform of Eq. (13) gives the following inhomogeneous second kind Volterra equation:

$$\psi_1(0,t) - \frac{1}{\Delta'_0} \int_0^t \psi_1(0,\tau) G(t-\tau) \mathrm{d}\tau = \frac{-\Delta'_s}{\Delta'_0} \psi_e(t), \tag{19}$$

where the kernel $G(t)$ is the inverse of the Laplace transform of $\Delta'_{in}(s)$ and is written as

$$G(t) = \frac{-4k}{3\tau_A} \left\{ \frac{\sqrt{\pi}}{2} \exp(\frac{t}{\tau_c}) + \sum_{n=1}^{\infty} \frac{\sqrt{n-1/4}}{n!} \times \Gamma(n-1/2) \exp(\frac{-t}{2\tau_n}) \sin(\frac{\sqrt{3}}{2}\frac{t}{\tau_n}) \right\} + \frac{k}{3\pi\tau_A} \int_0^{\infty} \times \sqrt{x} \left| \Gamma(ix - 1/4) \right|^2 \exp(-(4x)^{2/3} t/\tau_c - \pi x) \mathrm{d}x,$$

where

$$\tau_n = \frac{\tau_c}{(4n-1)^{2/3}}.$$

6. Summary and Discussion

We have improved the asymptotic matching in the boundary layer analysis of forced reconnection due to an externally imposed boundary perturbation, because it is shown that the asymptotic matching in previous works [1-4] is not appropriate and is valid only in the constant-ψ approximation. The appropriate asymptotic matching yields the exact reconnection process which reflects the effect of plasma inertia in the inner layer correctly.

It is shown that the characteristic time scales of forced reconnection in the initial evolution do not include the Sweet-Parker time scale deduced in previous works for island formation due to error fields [1,3,4] and for seed island formation [2]. The reconnected flux initially increases on the same time scale as the boundary perturbation, τ_e. Therefore the correct asymptotic matching conditions lead to a substantially different time scales for initial evolution of the forced reconnection due to a boundary perturbation from the ones in previous works. The correct asymptotic matching also yields the new feature of a surface current induced on the resonant surface. In the initial evolution, the surface current increases with $\Delta'(t) < 0$ so as to oppose the progress of the reconnection. In contrast, it was believed to be a typical feature of forced reconnection that $\Delta'(t) > 0$ and $\Delta'(t) \to +\infty$ at $t = 0$, in previous works [1,3,4].

These new results, which are the increase of the reconnected fluxes on the time scale without the Sweet-Parker time scale and the negative surface current, $\Delta'(t) < 0$, suggest a modification of the previous estimation for the $S = \tau_R/\tau_A$ scaling of the widths of the islands due to error fields [3] and of seed islands [2] in tokamak plasmas. This is because the previous analysis of the reconnection process subsequent to the initial evolution is based on an initial evolution characterized by an increase of the reconnected flux on the Sweet-Parker time scale and the positive surface current, $\Delta'(t) > 0$. Such a modification of the transition is expected to have a significant effect on the time scale of the reconnection and on the decay of the surface current in the nonlinear stage of island formation due to error fields [3] and seed island formation [2].

This modification is described by the subsequent evolution to the initial evolution. It will be obtained by numerically calculating the inverse of the Laplace transform of the reconnected flux by use of the integral equation.

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