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Plasma Shape Control and Equilibrium Beta Limit in Conventional Stellarators

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Abstract

The problem of plasma shaping and equilibrium β limit in conventional stellarators is considered analytically in traditional large-aspect-ratio approximation. The analysis allows to show explicitly the underlying physics. Besides physical insight, it gives useful estimates for potential increase of the equilibrium β limit and shows inherent limitations of plasma equilibrium control by external poloidal field in stellarators with shear. It is explained why in such systems the gain in equilibrium β limit due to plasma axisymmetric shaping only cannot be larger than 2.

Keywords:

conventional stellarator, MHD equilibrium, beta limit, plasma shaping

1. Introduction

Steady state fusion reactors must operate with high β plasma, β being the ratio of thermal plasma pressure to the magnetic field pressure. Stellarators with intrinsic steady state capabilities could be proper candidates for fusion reactor provided they would satisfy high β requirements. Till recently, the highest ever obtained volume-averaged β in stellarators was 2.14%, which was achieved in 1993 in the CHS torsatron [1,2]. The best operating helical device LHD will hopefully operate at β up to 5% [3], and an impressive step has been already done when LHD reached a new record value $\beta = 2.4\%$ [4]. These values of β are the result of careful optimization. However, economically reasonable reactor requires higher β. The problem of increasing MHD β limit in conventional stellarators still remains a challenge.

It is known that equilibrium β limit in conventional stellarators can be increased by elongating plasma cross-section [5]. This result was firstly obtained in numerical simulation of plasma equilibrium. Here we discuss a physical nature of the result and give some estimates showing the role of shear in the problem. Also, we consider separate and combined effect of quadrupole and octupole poloidal magnetic fields on stellarator magnetic configuration.

2. Basic Equations

Theory of plasma equilibrium in toroidal magnetic confinement systems is based on equations

$$\nabla p = \boldsymbol{j} \times \boldsymbol{B}, \tag{1}$$

$$\boldsymbol{j} = \operatorname{rot} \boldsymbol{B}, \ \operatorname{div} \boldsymbol{B} = 0. \tag{2}$$

Here p is the plasma pressure, $\boldsymbol{j}$ is the current density, and $\boldsymbol{B}$ is the magnetic field.

It follows from the first of Maxwell equations (2)

$$-\frac{2\pi}{r} j_t = \operatorname{div} \frac{\nabla(\psi - \psi_v)}{r^2}, \tag{3}$$

where j_t is the toroidal projection of the axisymmetric component of $\boldsymbol{j}$, ψ is the total poloidal flux embraced by a magnetic surface, ψ_v is the poloidal flux of the helical

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magnetic field, r is the radial coordinate; for more detail see [5].

In stellarators the helical field is small compared with toroidal magnetic field. It allows to use so-called stellarator expansion [5,6] for calculating j_t and ψ_v. For a plasma without net toroidal current

$$j_t = 2\pi r \frac{\mathrm{d}p}{\mathrm{d}\psi}(\Omega - \langle\Omega\rangle), \tag{4}$$

where

$$\Omega = 1 - \frac{R^2}{r^2} + \frac{\langle\tilde{B}^2\rangle_\zeta}{B_0^2}, \tag{5}$$

angular brackets <...> in (4) denote the averaging over the volume between neighboring surfaces ψ = const, R is the major radius, $\tilde{\boldsymbol{B}}$ is the helical magnetic field, $\langle ... \rangle_\zeta$ is the averaging over toroidal angle ζ, and B_0 is the toroidal field at the axis $r = R$. More information about these equations and expressions can be found in [5].

3. Equilibrium Beta Limit and Pfirsch-Schlüter Current

The current (4) is called a longitudinal equilibrium current, or a dipole or Pfirsch-Schlüter current. The difference $\Omega - \langle\Omega\rangle$ is always nonzero, at least due to toroidicity, therefore at nonzero pressure gradient we have some current j_t increasing with increasing pressure. It follows from (3) that larger j_t results in larger deviation of ψ from its vacuum value. The distortions of surfaces ψ = const are allowable only to some extent. The pressure at which the admissible level of deformations is reached would be an upper threshold due to equilibrium. Accordingly, β cannot be larger than some β_{eq}, which is called an equilibrium beta limit.

If it were possible to create a stellarator configuration where the plasma equilibrium would be realized with $j_t = 0$, this could be the ideal solution of the equilibrium problem. However, configurations without the Pfirsch-Schlüter current in conventional stellarators are found to be in that area of parameters where the plasma equilibrium is unstable. Moreover, this area lies so far from the acceptable ranges [7-9] that progress in that direction as a way of at least partial suppression of the Pfirsch-Schlüter current is hardly suitable for real experiment.

At smaller j_t the deviation of ψ from the vacuum solution to Eq. (3) is smaller, and, accordingly, the equilibrium limit β_{eq} is higher. It may seem that the reduction of $\Omega - \langle\Omega\rangle$ (studied theoretically in [7,8] and experimentally in Heliotron E [9]) is the only way of decreasing j_t. But this is not so. There is another multiplier in j_t. Let us show it in another form:

$$\frac{|\mathrm{d}p|}{|\mathrm{d}\psi|} = \frac{|\nabla p|}{|\nabla\psi|}. \tag{6}$$

It is clear that at a fixed minor radius, β is higher at larger allowable $|\nabla p|$. The presence of $|\nabla\psi|$ in the denominator of Eq. (6) suggests that for getting larger $|\nabla p|$ at given $|p'(\psi)|$ below some admissible level it is necessary to increase $|\nabla\psi|$.

4. Reduction of Pfirsch-Schlüter Current due to $|\nabla\psi|$ Increase

Apparently, the value $|\nabla\psi|$ could be increased by driving the net current through the plasma or by placing a conductor with a current inside the plasma. Mentioning this for completeness only, we are obliged, first of all, to consider the main, more realistic and attractive variant for stellarators: the control of ψ by the fields of external currents.

The effectiveness of such a control is determined by the relationship

$$\psi = \psi_v + \psi_{ext} + \psi_{pl}, \tag{7}$$

where ψ_{ext} is the poloidal flux due to external axisymmetric field, ψ_{pl} is the plasma-produced poloidal flux (in our case - produced by the Pfirsch-Schlüter current). Naturally, the poloidal flux ψ_{ext} of the external (vacuum) control field must satisfy the equation

$$\operatorname{div}\frac{\nabla\psi_{ext}}{r^2} = 0. \tag{8}$$

Therefore, the next condition must be valid for an arbitrary closed contour Γ on the plane (r, z):

$$\oint_\Gamma \frac{1}{r}\frac{\partial\psi_{ext}}{\partial n}\,dl = 0, \tag{9}$$

where n indicates the normal to Γ. This equality shows that the increase in $|\nabla\psi|$ owing to ψ_{ext} is not possible in all the directions from the magnetic axis, which is perfectly illustrated by the functions $f = 1, z, r^2, r^2z, r^4 - 4r^2z^2, \ldots$ being the particular solutions to Eq. (8).

Under such restrictions the choice should be unequivocally made for the benefit of the horizontal direction, along the r axis, which is the direction of the maximal inhomogeneity of the magnetic field. This inhomogeneity does not allow the transverse dimension of the plasma column to be increased: the "fatter" the magnetic surface, the larger $\Omega - \langle\Omega\rangle$ [see Eqs. (4) and (5)]. But in the vertical direction the toroidal field is homogeneous; therefore, the reduction in $|\nabla \psi|$ and the accompanying vertical elongation of magnetic surfaces in the vertical direction may be allowed.

Finally, using external poloidal field we can increase $|\nabla \psi|$ in horizontal direction, which is a desired goal, with simultaneous decrease in vertical direction, which is not desired, but not dangerous.

5. Reduction of Pfirsch-Schlüter Current by External Quadrupole Field

Using this method of reduction of the Pfirsch-Schlüter current, we have to ensure the increase in $|\psi-\psi_{axis}|$ on both sides from the circular axis of the configuration. A homogeneous vertical field $B_\perp \mathbf{e}_z$ cannot be used for this purpose, but a quadrupole field

$$\mathbf{B}_q = -\frac{B_2}{2b}\left[\nabla(\rho^2 \cos 2u)\times \mathbf{e}_\zeta\right], \qquad (10)$$

might be quite suitable. To see that, let us look at the expression

$$\psi = -\pi B_0 \rho^2 \left[\mu_0 + (\mu_b - \mu_0)\frac{\rho^2}{2b^2} + A\frac{B_2}{B_0}\cos 2u\right] \qquad (11)$$

for the function $\psi = \psi_v + \psi_q$ describing magnetic surfaces when quadrupole field (10) is imposed on a stellarator configuration with a parabolic vacuum rotational transform

$$\mu_h = \mu_0 + (\mu_b - \mu_0)\frac{\rho^2}{b^2}. \qquad (12)$$

Here B_2 is the strength of the field $\mathbf{B}_q$ at the distance b from the axis $r = R$, ρ, u are quasi-cylindrical coordinates related to this axis, μ_0 and μ_b are some constants, $A = R/b$ is the aspect ratio.

It follows from Eq. (11), where two first terms represent ψ_v, that in the equatorial plane

$$\frac{\psi(b)}{\psi_v(b)} = 1 + \frac{2B_2}{B_2^{cr} + B^*}, \qquad (13)$$

where

$$B_2^{cr} = \mu_0 \frac{B_0}{A}, \quad B^* = \mu_b \frac{B_0}{A}. \qquad (14)$$

At the geometrical axis $\psi = \psi_v = 0$, therefore, the ratio (13) can be considered as a quantitative measure of the change in $|\psi-\psi_{axis}|$ under the action of the quadrupole field (10). The larger the B_2, the better the result given by Eq. (13). However, at $B_2 = B_2^{cr}$ the internal separatrix appears, see Eq. (11), which then grows with increasing B_2 (for more explanations see [5]). At $B_2 = B_2^{cr}$ we have

$$\frac{\psi(b)}{\psi_v(b)} = 1 + \frac{2\mu_0}{\mu_b + \mu_0}, \qquad (15)$$

which is the upper limit for single-axis configurations $\psi = \text{const}$ described by (11). It is clear that

$$0 \le \frac{2\mu_0}{\mu_b + \mu_0} \le 1. \qquad (16)$$

Therefore, at $B_2 < B_2^{cr}$ the gain in $|\nabla \psi|$ at the expense of the quadrupole field cannot be larger than 2. At typical parameters of a stellarator this value is smaller, but 1.5-fold increase in $|\nabla \psi|$ is quite real when $\mu_0/\mu_b = 1/3$. Then it is possible also to expect a similar increase in β_{eq}.

6. Effect of Higher Harmonics of Poloidal Field

In general, an increase in $\psi(\rho) - \psi(0)$ on both sides of the axis $\rho = 0$ can be produced by external poloidal field of any even multipolarity. In a large aspect ratio approximation such a field is described by the poloidal flux function

$$\psi_n = -2\pi R \frac{B_n}{n b^{n-1}} \rho^n \cos nu \qquad (17)$$

with even n. The quadrupole field is the lowest appropriate harmonic ($n = 2$). If in addition there is an octupole field, Eq. (11) turns into

$$\psi = -\pi B_0 \rho^2 \left[\mu_0 + (\mu_b - \mu_0)\frac{\rho^2}{2b^2} + A\frac{B_2}{B_0}\cos 2u + A\frac{B_4}{B_0}\frac{\rho^2}{2b^2}\cos 4u\right]. \qquad (18)$$

Accordingly, in the equatorial plane ($\cos 2u = 1$)

$$\frac{\psi(b)}{\psi_v(b)} = 1 + \frac{2AB_2}{B_0(\mu_b + \mu_0)} + \frac{AB_4}{B_0(\mu_b + \mu_0)}, \qquad (19)$$

which is an extension of the expression (13). Since we need to increase this ratio, both B_2 and B_4 must be positive. As mentioned above, to avoid splitting of the axis under the action of the quadrupole field, we must keep B_2 below the critical value B_2^{cr} defined by Eq. (14). It can be shown by using (18) that under the mentioned conditions $0 < B_2 < B_2^{cr}$, $0 < B_4$ the octupole field does not break magnetic surfaces, if $B_4 < B_4^{cr}$, where

$$B_4^{cr} = (\mu_b - \mu_0)\frac{B_0}{A}. \qquad (20)$$

When the maximal admissible values $B_2 = B_2^{cr}$, $B_4 = B_4^{cr}$ are substituted into Eq. (19), it turns into

$$\frac{\psi(b)}{\psi_b(b)} = 2, \tag{21}$$

which is the upper limit for (19), if single-axis configurations are considered. Let us note that, contrary to (15), this ratio does not depend on μ_0/μ_b. This is the result of a proper choice of two parameters B_2 and B_4 in the model where the vacuum rotational transform is described by two parameters also, μ_0 and μ_b.

For configurations with $B_4^{cr} \neq 0$ the ratio (21) is larger than (15) because of additional contribution due to the octupole magnetic field. In (19) this contribution is $B_4/(2B_2)$ of that due to the quadrupole field. In (21) the octupole/quadrupole ratio is

$$\frac{B_4^{cr}}{2B_2^{cr}} = \frac{\mu_b - \mu_0}{2\mu_0} \tag{22}$$

This value is unity if $\mu_b = 3\mu_0$. Such a relation between rotational transform at the edge and at the axis is typical for stellarators. For example, it holds for CHS [1,2]. In LHD the ratio $\mu_b/3\mu_0$ is also close to 3 [3,4]. According to (22), in such devices both octupole and quadrupole poloidal fields are equally effective in increasing $\psi(\rho) - \psi(0)$ in the equatorial plane. Then, theoretically, in such stellarators, if the transverse size of the last closed magnetic surface could be kept unchanged, it would be possible to increase β_{eq} by one half of the "ordinary" limit applying the quadrupole field and to get the same additional increase in β_{eq} due to the octupole field.

7. Shear and Shaping

It is clear from the above analysis that the main characteristic determining the reaction of the system on the shaping magnetic field is the rotational transform. In our case it is modeled by the parabola (12), which makes possible to treat shear without restrictions. Following the scheme, one can easily consider a case with more complicated dependence $\mu_h(\rho)$, but the model (12) is quite sufficient to analyze the role of shear.

The main parameter, characterizing the vacuum shear at the absence of shaping fields, is μ_0/μ_b, which is the ratio of the vacuum rotational transform at the axis to that at the edge. It first appears in (15). In the case of strong shear the ratio μ_0/μ_b is small, and, accordingly, small is the possible gain in β_{eq} due to the quadrupole field. On the other hand, small μ_0/μ_b results in large ratio (22). It means that in this case the octupole field would be much more effective than quadrupole. This can be shown explicitly by

$$\frac{\psi(b)}{\psi_v(b)} = 1 + \frac{\mu_b - \mu_0}{\mu_b + \mu_0} \tag{23}$$

for the case $B_2 = 0$, $B_4 = B_4^{cr}$, which should be compared with (15) for $B_2 = B_2^{cr}$, $B_4 = 0$.

However, if shear is large (the difference $\mu_b - \mu_0$ is small), the situation is changed to the opposite: efficiency of the quadrupole field becomes much higher, while the octupole field splits magnetic surfaces at small B_4.

The mentioned case $\mu_b = 3\mu_0$ is exactly intermediate, when octupole and quadrupole poloidal fields may give an equal increase in $\psi(b)/\psi_v(b)$.

8. Summary

This analysis clearly shows how and why it would be possible to control the magnitude of the current without shifting the plasma. Because of the utmost simplicity of the given arguments, the conclusion that the positive effect of vertical elongation is possible in stellarators (though only up to a certain level) is beyond doubt.

It is shown that the optimal combination of the shaping fields for larger gain in β_{eq} depends on the shear of initial vacuum configuration. If shear is small, the quadrupole field would be much more effective than the octupole for getting higher β_{eq}, and vice versa. Stellarators with $\mu_b = 3\mu_0$ represent an intermediate case, when octupole and quadrupole poloidal fields may give an equal integral effect.

Our discussion is concentrated on one mechanism only, increase of $\psi(\rho) - \psi(0)$ in the radial direction. The real gain in β_{eq} will depend also on the plasma pressure profile obtained in a device. For larger β_{eq} we need larger $|\nabla\psi|$ in the region of larger pressure gradient. The octupole field can give large $|\nabla\psi|$ only at the periphery. So its efficiency will be determined also by such factors as pressure gradient at the periphery and the quality of the magnetic surfaces near the edge. The same is even more valid for fields of higher multipolarity, and they hardly can be useful for increasing β_{eq} in conventional stellarators.

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