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Bounce-Resonant Wave-Particle Interactions in Tokamaks with Elliptic Magnetic Surfaces

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Abstract

The longitudinal permittivity elements of a toroidal plasma with elliptic magnetic surfaces are evaluated by solving the drift kinetic equation as a boundary-value problem for waves in the frequency range much larger than the drift frequency. The quadratic corrections over the inverse tokamak aspect ratio are taken into account in order to approximate the equilibrium magnetic field. The separate contributions of untrapped, *t*-trapped and *d*-trapped particles to the longitudinal permittivity are written by the summation of bounce-resonant terms including the well-known plasma dispersion function.

Keywords:
bounce resonance, elongated tokamak, longitudinal permittivity

1. Introduction

The problems of plasma heating and current drive in large-size tokamaks by using the radio-frequency waves have stimulated renewed interest in the study of the dielectric properties of a magnetized toroidal plasma taking into account the ellipticity/triangularity of the magnetic surfaces and the bounce-resonant wave-particle interactions there. In this paper, we evaluate the longitudinal permittivity elements of a collisionless toroidal plasma with elliptic magnetic surfaces. In contrast to Refs. [1-3] related to it, we describe the untrapped and three groups of trapped particles in the general case, when all these particle groups exist at the considered magnetic surfaces, accounting the quadratic corrections over the inverse tokamak aspect ratio. To solve the drift kinetic equation for the perturbed distribution function $f = f(\rho, \theta, v_{\|}, v_{\perp})\exp(-i\omega t + in\phi)$ of charged particles we introduce the new variables (r, θ') instead of the quasi-toroidal coordinates (ρ, θ)

$$r = \rho\sqrt{\frac{a^2}{b^2}\sin^2\theta + \cos^2\theta}\,, \quad \theta' = \arctan\left(\frac{a}{b}\tan\theta\right),$$

and the "magnetic moment" $\mu = \sin^2\gamma\sqrt{H_{\phi 0}^2 + H_{\theta 0}^2}/H$ instead of the pitch-angle $\gamma\,(v_{\|} = v\cos\gamma,\ v_{\perp} = v\sin\gamma)$. Here, $H_{\phi 0}$ and $H_{\theta 0}$ are the toroidal and poloidal projections of an equilibrium magnetic field, $\boldsymbol{H}$, for a given magnetic surface, at the points $\theta = \pm\pi/2$. The modulus of $\boldsymbol{H}$ is

$$H = \sqrt{H_{\phi 0}^2 + H_{\theta 0}^2}\,\frac{\sqrt{1+\lambda\cos^2\theta'}}{1+\varepsilon\cos\theta'} \approx$$
$$\approx \sqrt{H_{\phi 0}^2 + H_{\theta 0}^2}\left(1 - \varepsilon\cos\theta' + \frac{\varepsilon^2}{1+\varepsilon}\cos^2\theta' + \frac{\lambda}{2}\cos^2\theta'\right).$$

Here

$$\varepsilon = \frac{r}{R}, \quad q = \varepsilon\frac{h_\phi}{h_\theta},$$
$$\lambda = h_\theta^2\left(\frac{b^2}{a^2} - 1\right) = \frac{\varepsilon^2}{q^2+\varepsilon^2}\left(\frac{b^2}{a^2} - 1\right),$$
$$h_\theta = \frac{H_{\theta 0}}{\sqrt{H_{\phi 0}^2 + H_{\theta 0}^2}}, \quad h_\phi = \frac{H_{\phi 0}}{\sqrt{H_{\phi 0}^2 + H_{\theta 0}^2}},$$

where R is the major tokamak radius; b and a are the

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major and minor semiaxes of the elliptic cross-section of the external magnetic surface. In this model, all magnetic surfaces are similar each other with the same elongation equal to b/a. Assuming that ε and λ are small, the perturbed distribution function can be presented in the form

$$f(\tilde{\theta}) = \sum_s \tilde{f}_s(\tilde{\theta}) \exp\left[inq \sin\tilde{\theta}\left(\varepsilon + \zeta\cos\tilde{\theta}\right)\right],$$

where $\tilde{\theta} = \theta' + 0.125\lambda \sin 2\theta'$ and $\zeta = 0.25\lambda - 0.5\varepsilon^2/(1+\varepsilon)$.

As a result, for $\tilde{f}_s$ we have

$$\frac{\partial \tilde{f}_s}{\partial \tilde{\theta}} + i\phi_s(\tilde{\theta})\tilde{f}_s = Q(\tilde{\theta}), \tag{1}$$

where

$$\phi_s(\eta) = nq\left(1 + \frac{0.5\varepsilon^2}{1+\varepsilon}\right) - \frac{s\omega(1+0.25\lambda)}{k_0 v\sqrt{1-\mu(1-\varepsilon\cos\eta + \nu\cos^2\eta)}},$$

$$Q(\eta) = \frac{eE_{\parallel}(\eta)}{k_0 T}\left(1+\frac{\lambda}{4}\right)\exp\left[-inq\sin\eta\,(\varepsilon + \zeta\cos\eta)\right] F_0,$$

$$\nu = \frac{\lambda}{2} + \frac{\varepsilon^2}{1+\varepsilon}, \quad k_0 = \frac{h_\theta}{r}, \quad F_0 = \frac{N_0}{(\pi v_T^2)^{1.5}}\exp\left(-\frac{v^2}{v_T^2}\right).$$

The steady-state distribution function F_0 is given as maxwellian, where N_0 is the particle density, and $v_T = \sqrt{2T/M}$ is the thermal velocity of particles with temperature T and mass M. By indexes $s = \pm 1$ for f_s we distinguish the perturbed distribution functions over positive and negative values of parallel velocity

$$v_{\parallel} = s v\sqrt{1-\mu(1-\varepsilon\cos\tilde{\theta} + \nu\cos^2\tilde{\theta})}$$

relative to $\boldsymbol{H}$. After solving Eq. (1), the longitudinal component of the current density $j_{\parallel} = \boldsymbol{j}\cdot\boldsymbol{H}/H$ can be expressed by

$$j_{\parallel}(\tilde{\theta}) = \pi e\,\frac{\exp\left[inq\sin\tilde{\theta}\left(\varepsilon+\zeta\cos\tilde{\theta}\right)\right]}{(1-\varepsilon\cos\tilde{\theta}+\nu\cos^2\tilde{\theta})^{-1}} \times \sum_s^{\pm 1} s\int_0^{\infty} v^3 \int_0^{(1-\varepsilon\cos\tilde{\theta}+\nu\cos^2\tilde{\theta})^{-1}} \tilde{f}_s(\tilde{\theta},\mu,v)\,d\mu\, dv. \tag{2}$$

Depending on the parameters μ and $\tilde{\theta}$ the phase volume of plasma particles should be split in the phase volumes of untrapped and trapped particles. This separation can be done by the condition $v_{\parallel}(\mu, \tilde{\theta}) = 0$. In contrast to the tokamak with circular magnetic surfaces, for a plasma torus with elliptic magnetic surfaces, there is possible the existence [1-3] of two additional groups of the trapped particles at such magnetic surfaces where $\varepsilon < \lambda$. And really, $\mu(\tilde{\theta})$ as the solution of $v_{\parallel}(\mu, \tilde{\theta}) = 0$, in the general case, has five extremums where $\tilde{\theta} = \pm\pi$, 0, $\pm$ arccos (ε/λ). To have the full physical picture, we derive the contributions of all four groups of plasma particles to the longitudinal permittivity of a toroidal plasma with elliptic magnetic surfaces. As a result, the phase volumes of the different groups of plasma particles are defined by the following inequalities:

$0 \le \mu \le \mu_u$, $\quad -\pi \le \tilde{\theta} \le \pi$ –for untrapped particles (zone 1);

$\mu_u \le \mu \le \mu_t$, $\quad -\theta_t \le \tilde{\theta} \le \theta_t$ –for t-trapped particles, (zone 2);

$\mu_t \le \mu \le \mu_d$, $\quad -\theta_t \le \tilde{\theta} \le -\theta_d$ –for d-trapped particles, (zone 3);

$\mu_t \le \mu \le \mu_d$, $\quad \theta_d \le \tilde{\theta} \le \theta_t$ –for d-trapped particles, (zone 4),

where the reflection points θ_t and θ_d for the trapped particles are defined by the relationships

$$\cos\theta_t = \frac{\varepsilon}{2\nu}\left[1 - \sqrt{1 + \frac{4\nu(1-\mu)}{\varepsilon^2\mu}}\right],$$

$$\cos\theta_d = \frac{\varepsilon}{2\nu}\left[1 + \sqrt{1 + \frac{4\nu(1-\mu)}{\varepsilon^2\mu}}\right],$$

$$\mu_u = \frac{1+\varepsilon}{1+(2\varepsilon+0.5\lambda)(1+\varepsilon)}, \quad \mu_t = \frac{1+\varepsilon}{1+0.5\lambda(1+\varepsilon)},$$

$$\mu_d = \frac{2\lambda(1+\varepsilon)+4\varepsilon^2}{2\lambda(1+\varepsilon)+3\varepsilon^2-\varepsilon^3}.$$

The solution of Eq. (1) can be found by the specific boundary condition for the trapped and untrapped particles. For untrapped particles (zone 1), we use the periodicity of $\tilde{f}_s$ over θ. The boundary condition for the t-trapped (zone 2) and d-trapped (zones 3 and 4) particles is the continuity of $\tilde{f}_s$ at the corresponding stop-points (local mirrors or reflection points) where the parallel velocity is equal to zero. As usual, the equality of denomenators of $\tilde{f}_s$ to zero gives us the bounce resonance conditions of an effective wave-particle interactions. Using Eq. (2), it is possible to evaluate the separate contributions of all groups of plasma particles to the longitudinal current density component.

2. Longitudinal Permittivity Elements

The dielectric tensor elements can be derived after the Fourier expansions of current density and electric field over the angle $\tilde{\theta}$:

$$j_{\parallel}(\tilde{\theta})(1+\varepsilon\cos\tilde{\theta} - \nu\cos^2\tilde{\theta}) = \sum_{m=-\infty}^{+\infty} j_{\parallel}^{(m)}\exp(im\tilde{\theta}),$$

$$E_{\|}(\tilde{\theta}) = \sum_{m'=-\infty}^{+\infty} E_{\|}^{(m')} \exp(\mathrm{i}m'\tilde{\theta}),$$

Thus, the harmonics $j_{\|}^{(m)}$ and $E_{\|}^{(m')}$ are connected by the longitudinal permittivity elements:

$$\frac{4\pi\mathrm{i}}{\omega} j_{\|}^{(m)} = \sum_{m'=-\infty}^{+\infty} \varepsilon_{\|}^{m,m'} E_{\|}^{(m')} = \sum_{m'=-\infty}^{+\infty} \left(\varepsilon_{\|,u}^{m,m'} + \varepsilon_{\|,t}^{m,m'} + \varepsilon_{\|,d}^{m,m'} \right) E_{\|}^{(m')},$$

where $\varepsilon_{\|,u}^{m,m'}$, $\varepsilon_{\|,t}^{m,m'}$, and $\varepsilon_{\|,d}^{m,m'}$ are the separate contributions of untrapped, t-trapped and d-trapped particles, respectively, to the longitudinal permittivity (the details see in Refs. [1,3]).

The contribution of untrapped particles to the longitudinal permittivity is

$$\varepsilon_{\|,u}^{m,m'} = \frac{2\omega_{po}^2}{k_0^2 \upsilon_T^2 \pi^3} \sum_{p}^{+\infty} \int_0^{\mu_u} \frac{(1+0.5\lambda) K(\kappa)}{(m+nq+p)^2} \frac{\left[1 + 2u_p^2 + 2i\sqrt{\pi} u_p^3 W(u_p)\right] A_p^m A_{p+m-m'}^{m'}}{\sqrt{1-\mu(1-\nu) + \sqrt{\varepsilon^2\mu^2 + 4\nu\mu(1-\mu)}}} \mathrm{d}\mu, \quad (3)$$

Here, we used the following definitions:

$$u_p = \frac{\omega(1+0.25\lambda)\sqrt{2\kappa}K(\kappa)}{k_0 \left| m+nq+p \right| \upsilon_T \pi [\varepsilon^2\mu^2 + 4\nu\mu(1-\mu)]^{0.25}},$$

$$\omega_{po}^2 = \frac{4\pi N_0 e^2}{M},$$

$$\kappa = \frac{2\sqrt{\varepsilon^2\mu^2 + 4\nu\mu(1-\mu)}}{1-(1-\nu)\mu + \sqrt{\varepsilon^2\mu^2 + 4\nu\mu(1-\mu)}},$$

$$A_p^m(\kappa) = \int_{-K(\kappa)}^{K(\kappa)} \exp\Big[\mathrm{i}\frac{p\pi\omega}{K(\kappa)} - 2\mathrm{i}(m+nq)\chi_u(\kappa,w) + 2\mathrm{i}nq\psi_u(\kappa,w)\Big] \frac{\sqrt{1+\beta}\,\mathrm{dn}(\kappa,w)}{1+\beta\,\mathrm{cn}^2(\kappa,w)} \mathrm{d}w,$$

$$\beta = \frac{4\nu}{\varepsilon - 2\nu + \sqrt{\varepsilon^2 + 4\nu(1-\mu)/\mu}},$$

$$K(\kappa) = \int_0^{\pi/2} \frac{\mathrm{d}\eta}{\sqrt{1-\kappa\sin^2\eta}},$$

$$\chi_u(\kappa,w) = \arctan\left(\frac{1}{\sqrt{1+\beta}} \frac{\mathrm{sn}(\kappa,w)}{\mathrm{cn}(\kappa,w)} \right) - \frac{\pi w}{2K(\kappa)},$$

$$\psi_u(\kappa,w) = \sqrt{1+\beta}\, \frac{\mathrm{sn}(\kappa,w)\,\mathrm{cn}(\kappa,w)}{1+\beta\,\mathrm{cn}^2(\kappa,w)} \left[\varepsilon + \zeta \frac{(2+\beta)\,\mathrm{cn}^2(\kappa,w) - 1}{1+\beta\,\mathrm{cn}^2(\kappa,w)} \right],$$

where $\mathrm{sn}(\kappa, w)$, $\mathrm{cn}(\kappa, w)$, $\mathrm{dn}(\kappa, w)$ are the Jacobi elliptic functions [4,5].

The contribution of t-trapped particles to the longitudinal permittivity is

$$\varepsilon_{\|,t}^{m,m'} = \frac{\sqrt{8}\,\omega_{po}^2}{k_0^2 \upsilon_T^2 \pi^3} \sum_{p=1}^{\infty} \int_{\mu_u}^{\mu_t} \frac{(1+0.5\lambda) K(\kappa^{-1}) B_p^m B_p^{m'}}{p^2 [\varepsilon^2\mu^2 + 4\nu\mu(1-\mu)]^{1/4}} \times \left[1 + 2\upsilon_p^2 + 2i\sqrt{\pi}\upsilon_p^3 W(\upsilon_p)\right] \mathrm{d}\mu, \quad (4)$$

$$\upsilon_p = \frac{2\sqrt{2}\,\omega(1+0.25\lambda) K(\kappa^{-1})}{k_0 p \upsilon_T \pi [\varepsilon^2\mu^2 + 4\nu\mu(1-\mu)]^{1/4}},$$

$$B_p^m(\kappa) = \sqrt{\frac{1+\beta}{\kappa}} \int_{-2K(\kappa^{-1})}^{2K(\kappa^{-1})} \exp\Big[\frac{\mathrm{i}p\pi w}{2K(\kappa^{-1})} - 2\mathrm{i}(m+nq)\chi_t(\kappa,w) + 2\mathrm{i}nq\psi_t(\kappa,w)\Big] \frac{\mathrm{cn}(\kappa^{-1},w)\,\mathrm{d}w}{1+\beta\,\mathrm{dn}^2(\kappa^{-1},w)},$$

$$\chi_t(\kappa,w) = \arcsin\left(\frac{\kappa^{-1/2}\,\mathrm{sn}(\kappa^{-1},w)}{\sqrt{1+\beta\,\mathrm{dn}^2(\kappa^{-1},w)}} \right),$$

$$\psi_t = \sqrt{\frac{1+\beta}{\kappa}}\, \frac{\mathrm{sn}(\kappa^{-1},w)\,\mathrm{dn}(\kappa^{-1},w)}{1+\beta\,\mathrm{dn}^2(\kappa^{-1},w)} \left[\varepsilon + \zeta \frac{(2+\beta)\,\mathrm{dn}^2(\kappa^{-1},w) - 1}{1+\beta\,\mathrm{dn}^2(\kappa^{-1},w)} \right].$$

The contribution of d-trapped particles to the longitudinal permittivity is

$$\varepsilon_{\|,d}^{m,m'} = \frac{2\omega_{po}^2(1+0.5\lambda)}{k_0^2 \upsilon_T^2 \pi^3} \sum_{p=1}^{\infty} \frac{1}{p^2} \int_{\mu_t}^{\mu_d} \left[1 + 2\tau_p^2 + 2i\sqrt{\pi}\tau_p^3 W(\tau_p)\right] \times \frac{\kappa^2 K(\kappa) \left[D_p^m \left(D_p^{m'}\right)^* + D_p^{m'} \left(D_p^m\right)^* \right]}{\sqrt{1-(1-\nu)\mu + \sqrt{\varepsilon^2\mu^2 + 4\nu\mu(1-\mu)}}} \mathrm{d}\mu, \quad (5)$$

where

$$\tau_p = \frac{\omega(1+0.25\lambda)\sqrt{2\kappa}K(\kappa)}{k_0 p \upsilon_T \pi [\varepsilon^2\mu^2 + 4\nu\mu(1-\mu)]^{1/4}},$$

$$D_p^m(\kappa) = \int_{-K(\kappa)}^{K(\kappa)} \exp\Big[\mathrm{i}\frac{p\pi w}{K(\kappa)} - \mathrm{i}(m+nq)\chi_d(\kappa,w) + \mathrm{i}nq\psi_d(\kappa,w)\Big] \frac{\sqrt{\delta}\,\mathrm{cn}(\kappa,w)\,\mathrm{sn}(\kappa,w)}{\delta + \mathrm{dn}^2(\kappa,w)} \mathrm{d}w,$$

$$\delta = \frac{2\nu - \varepsilon - \sqrt{\varepsilon^2 + 4\nu(1-\mu)/\mu}}{2\nu + \varepsilon + \sqrt{\varepsilon^2 + 4\nu(1-\mu)/\mu}},$$

$$\chi_d(\kappa, w) = \arccos\left[\frac{\mathrm{dn}^2(\kappa, w) - \delta}{\mathrm{dn}^2(\kappa, w) + \delta}\right],$$

$$\psi_d(\kappa, w) = \frac{2\sqrt{\delta}\,\mathrm{dn}(\kappa, w)}{\mathrm{dn}^2(\kappa, w) + \delta}\left[\varepsilon + \zeta\frac{\mathrm{dn}^2(\kappa, w) - \delta}{\mathrm{dn}^2(\kappa, w) + \delta}\right].$$

The phase coefficients A_p^m, B_p^m, and D_p^m do not depend on the wave frequency that allowed us to obtain the final expressions for $\varepsilon_{\parallel,u}^{m,m'}$, $\varepsilon_{\parallel,t}^{m,m'}$ and $\varepsilon_{\parallel,d}^{m,m'}$ by the plasma dispersion function $W(z) = \exp(-z^2)\,[1 + (2i/\sqrt{\pi})\int_0^z \exp(t^2)dt]$. As a result, the bounce-resonant conditions can be presented by:

$u_p = 1$ for the untrapped particles, where the numbers of bounce resonances are $p = 0, \pm1, \pm2, \ldots$;
$\upsilon p = 1,\ p = 1, 2, 3, \ldots$ for the t-trapped particles;
and $\tau_p = 1,\ p = 1, 2, 3, \ldots$ for the d-trapped particles.

3. Conclusions

The basic feature of elongated tokamaks is a possible existence of three groups of the trapped particles. Together with the usual (for tokamaks with circular magnetic surfaces) untrapped and t-trapped particles, two additional groups of d-trapped particles can appear at such elliptic magnetic surfaces where $\lambda > \varepsilon$. In other terms this criterion can be rewritten as

$$b/a > \sqrt{1 + \varepsilon + q^2/\varepsilon}\,.$$

The d-particles are located near the points $\theta = \pm\arccos(\varepsilon/\lambda)$. The trajectories of untrapped, t-trapped and d-trapped particles are entirely different. The energetic d-particles (e.g., the fusion-born alpha particles) can excite the toroidicity-induced [6] and ellipticity-induced [7] Alfvén eigenmodes. Moreover, these new nature particles can affect substantially the current drive generation and transport processes. Of course, choosing the level and profile of an equilibrium current (or by given the suitable radial structure of a tokamak safety factor, q), it is possible to realize the experimental regimes when the d-particles are absent into the plasma. As easily to see, the criterion $\lambda > \varepsilon$ is not satisfied in tokamaks with small elongation $b/a < 2$ and $q > 1$. This means that there are no d-trapped particles in the JET (as well ITER) plasmas.

The bounce resonance conditions for each of four groups of plasma particles (both the electrons and ions) are derived by solving the drift-kinetic equation. The longitudinal permittivity elements, Eqs. (3)-(5), are valid in the wide range of wave frequencies, wave numbers, and plasma parameters in the tokamaks with an elliptic cross-section of the magnetic surfaces.

Since the drift kinetic equation is solved as a boundary-value problem, the $\varepsilon_{\parallel,u,t,d}^{m,m'}$-elements are suitable to study the wave processes with a regular frequency such as the wave propagation, plasma heating and current drive, when the wave frequency is given by the antenna/generator system. The expressions (3)-(4) have a natural limit to the corresponding results [3,5] for plasmas with circular magnetic surfaces if $b = a$ and ν, $\zeta \to 0$. Note, the quadratic corrections over $\varepsilon = r/R$ are important to describe more correctly the finite-ε effects in tokamaks with $a/R \sim 1/3$ and $b \geq a$. The concrete computer calculations of $\varepsilon_{\parallel}^{m,m'}$-elements evaluated in this paper have been done in Ref. [8], where the kinetic Alfvén wave dissipation (by the trapped/untrapped electrons) is analyzed in an elongated tokamak with the main JET-parameters.

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