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Self-Organization and Beltrami Conditions

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Abstract

A plasma flow induces mixing of magnetic flux, and the length-scale cascades toward a small scale, resulting in amplification of the magnetic field. If the enhanced Lorentz force becomes to dominate the dynamics, its back-reaction should be taken into account. Once a Beltrami field is generated, it can stay stable and stationary against the stretching effect of the plasma flow. The Beltrami condition, which reads as the force-free condition, imposes a lower bound for the length-scale, avoiding the scale reduction down to the resistive regime. A statistical model of current filaments accounts well for the spectral structure of intermittent fluctuations of local currents in experiments.

Keywords:

Beltrami field, intermittency, filamentary structures

1. Introduction

The Beltrami condition describes the alignment of the vector field and its own vorticity. In many different convective nonlinear systems, the Beltrami condition plays an important role in characterizing self-organized structures [1,2]. A plasma flow induces mixing of magnetic flux, and the length-scale cascades toward a small scale, resulting in amplification of the magnetic field. If the enhanced Lorentz force becomes to dominate the dynamics, its back-reaction must be taken into account. The Beltrami condition, which reads as the magnetic force-free condition, must apply to slow motion of a strongly magnetized plasma, i.e., the magnetic field $\boldsymbol{B}$ must satisfy $\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}$. When λ is a constant over a localized current, this relation imposes a lower bound for the length-scale. This bound avoids scale reduction down to the resistive regime, and extends the life-time of the amplified magnetic field [3]. In laboratory experiments, we observe that local currents in a turbulent plasma fluctuate intermittently, suggesting that the current density has a strongly inhomogeneous distribution. A statistical model of current filaments accounts well for the spectral structure of the fluctuations. We consider a Boltzmann distribution of the size of the filaments that maximizes the entropy for an ensemble defined by the total current. An interesting assertion is that the time series produced by a random-motion model of such filaments generates a power-law spectra which agrees well with the observation [4].

2. Beltrami Condition

Due to the complex mixing process, the current tends to concentrate in small volumes, which may be disconnected. When the sectional length-scale of such a volume becomes small enough, the Lorentz force dominates. We consider such a "clump" of the magnetic field that is created by the kinematic dynamo process. For an arbitrary shape of the clump, the Beltrami condition $\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}$ can be satisfied with a constant λ such that $\mu_{-1} < \lambda < \mu_1$, where $\mu_\pm$ are the eigenvalues of the self-adjoint part of the curl operator in the domain Ω of the clump [5]. This Ω is assumed to be multiply connected (genus *geq* 1) and bounded. The eigenvalues

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are shown to be countable, and are numbered as $\cdots \le \mu_{-2} \le \mu_{-1} < 0 < \mu_1 \le \mu_2 \le \cdots$. The magnitudes of $\mu_\pm$ are of the order of the cross-sectional length scale of Ω. The force-free magnetic clump in the range of $\mu_{-1} < \lambda < \mu_1$ is stable against internal kink modes, while the stability of external modes (deformation of Ω) depends on exterior conditions. Once it is created, the internal magnetic field is maximally smooth, and no smaller internal structure can develop in the clump.

3. Statistical Model of Filaments

While each filamentary flux tube, such as a Beltrami field, can be locally stable, their movement and mutual interactions are very complicated, and hence, they invoke a statistical mechanical treatment. Let us consider a system of current filaments. Each filament is denoted by an index m ($m = 1, \cdots, N$; N is the total number of filaments). We specify the current density J_m on each filament. We can invoke the analogy of the standard statistical mechanics of particles, where J_m parallels the energy level of the eigenstate m. The cross section σ_m of the filament m is the statistical variable of the present model (σ_m may be regarded as the number of particles allocated to the eigenstate m). Each filament has the current of $I_m = J_m \sigma_m$. The total current is given by $I = \Sigma_{m=1}^{N} I_m$. A micro-state is characterized by specifying $\ell \equiv \{\sigma_1, \sigma_2, \cdots, \sigma_N\}$. The probability of a micro-state ℓ is denoted by $p(\ell)$.

The expectation value of the current is given by $\langle I \rangle = \Sigma_\ell p(\ell)\, I(\ell)$, where $I(\ell)$ is the current of the micro-state ℓ. Maximizing the Shannon entropy $S = -\Sigma_\ell p(\ell) \ln p(\ell)$ for the above-mentioned canonical ensemble, we obtain $p(\ell) = \exp(-\beta \Sigma_m J_m \sigma_m)/Z$, where $Z \equiv \exp(\alpha) = \Sigma_\ell \exp(-\beta I(\ell))$ Z is the partition function of the canonical distribution.

We assume that the probability for a filament "m" to have a cross-section σ_m (denoted by $p_m(\sigma_m)$) is independent to the all other filaments. Then, $p(\sigma_1, \sigma_2, \cdots, \sigma_N)$ can be broken down into the simple product of $p_m(\sigma_m)$ ($m = 1, \cdots, N$), and we obtain the "Boltzmann distribution"

$$p_m(\sigma_m) = \frac{\exp(-\beta J_m \sigma_m)}{Z_m},$$
$$Z_m = \sum_{\sigma_m} (-\beta J_m \sigma_m). \qquad (1)$$

Using the probability distribution (1), we generate the time series of the current density $j(t)$ that is measured at a fixed position. Note that the probability of measuring the filament m differs from $p_m(\sigma_m)$ (the probability of the filament m to have a cross section σ_m). The probability of measuring a filament m, at a fixed observation point, is proportional to σ_m.

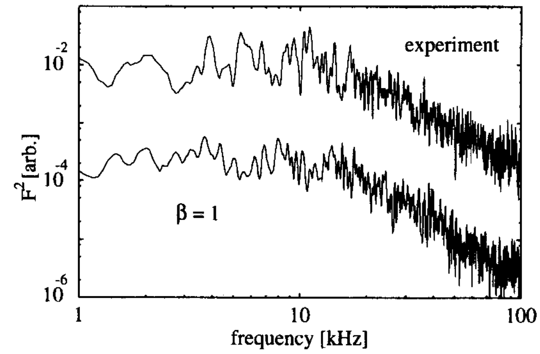

Fig. 1 Power spectrum of $j(t)$ produced by the statistical model, in comparison with the experimental spectrum. We apply different normalizations to both spectra to plot them at different vertical positions.

Figure 1 shows the frequency spectrum of $j(t)$ generated by the theory, in comparison with the experimental spectrum. We observe that the theoretical spectrum agrees well with the experimental one. The spectrum of $j(t)$ has two distinct frequency ranges; the spectrum in the low frequency range ($f < f_c \approx 20$kHz) is "white", while in the high frequency range ($f > f_c$), the spectrum is the $1/f$ spectrum. In the theory, the critical frequency f_c is given by $f_c = 1/(\pi \Delta t_c) = v/(2\pi\rho_c)$, where ρ_c is the average radius, Δt_c is the average duration of a filament, and v is the average speed (random walk) of filaments. The two different frequency ranges of the spectrum can be explained as follows. In a long time scale ($|t_1 - t_2| > \Delta t_c$), $j(t_1)$ and $j(t_2)$ come from different randomly-selected filaments. Therefore, we obtain a white spectrum. Agreement of our theoretical result with the experimental spectrum in this frequency range justifies our assumption of random movement of filaments.

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