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Generalized Magnetic Coordinates

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Abstract

The method to construct the generalized magnetic coordinate system is presented, which describe the magnetic field with and without magnetic surfaces. The method is applied to a simple analytic field.

Keywords:

generalized magnetic coordinate, GMC, magnetic flux coordinate, magnetic surface, ABC magnetic field, Fourier mode

1. Introduction

The magnetic coordinates[1,2] are widely used tools in the study of the MHD equilibrium and stability in the toroidal plasma, as well as the kinetic equations. The magnetic coordinates, or the flux coordinates, are closely related to the existence of the magnetic surface.

Unfortunately, the good magnetic surfaces exist only in the limited region of torus and even inside the outermost magnetic surface there might exist complicated island structure. In such cases, the utilization of the conventional magnetic flux coordinates is not expected. Some efforts to generalize the flux coordinates to general magnetic configuration are reported[3].

The Generalized Magnetic Coordinates (GMC) are the new one to supplement the flux coordinates system adequate to treat the general magnetic configurations. In GMC (ξ, η, ζ) the magnetic field is expressed in the form

$$\boldsymbol{B} = \nabla \Psi(\xi, \eta, \zeta) \times \nabla \zeta + H^{\zeta}(\xi, \eta) \nabla \xi \times \nabla \eta. \quad (1)$$

When the good magnetic surface exists, Ψ becomes independent of ζ and $\Psi(\xi, \eta) = \text{const.}$ is the magnetic surface. The ζ-dependent part of Ψ corresponds to the destruction of the magnetic surface.

Such a coordinate system is introduced in Ref.[4], and it is shown that it exists even in the outer region of the last closed surface. In this paper the general method to obtain the such coordinates is presented; and the method is applied for the simple magnetic field.

2. Construction of GMC

We shall consider a curvilinear coordinate system (ξ, η, ζ), ζ being the angle variable corresponding to the toroidal direction. The magnetic induction densities, the product of the contravariant component of the magnetic field multiplied by Jacobian $\sqrt{g}$, can be expressed in terms of the vector potential as

$$H^{\xi} = \frac{\partial A_{\zeta}}{\partial \eta} - \frac{\partial A_{\eta}}{\partial \zeta}, \quad H^{\eta} = \frac{\partial A_{\xi}}{\partial \zeta} - \frac{\partial A_{\zeta}}{\partial \xi},$$

$$H^{\zeta} = \frac{\partial A_{\eta}}{\partial \xi} - \frac{\partial A_{\xi}}{\partial \eta}. \quad (2)$$

We shall introduce the notations

$$\bar{A} \equiv \oint A \mathrm{d}\zeta / \oint \mathrm{d}\zeta, \quad \tilde{A} \equiv A - \bar{A}. \quad (3)$$

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The GMC is characterized as

1) H^ζ does not depend on ζ,

2) The variation of A_ζ in ζ direction is minimal.

These two conditions are represented as

$$\oint\left|\tilde{H}^\zeta\right|^2 \mathrm{d}\zeta = 0, \quad \delta\oint\left|\tilde{A}_\zeta\right|^2 \mathrm{d}\zeta = 0. \tag{4}$$

For simplicity we introduce the vector-like notations; *i.e.*, we put

$$\begin{aligned} &\boldsymbol{H} \equiv (H^\xi, H^\eta, H^\zeta), \quad \boldsymbol{A} \equiv (A_\xi, A_\eta, A_\zeta), \\ &\boldsymbol{e}_3(0,0,1), \quad \boldsymbol{x} \equiv (\xi, \eta, \zeta), \\ &\mathrm{grad} \equiv \left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right), \end{aligned} \tag{5}$$

and treat them as if they are 3-dimensional vectors.

If we introduce a time-like parameter τ and consider the continuous path from an initial state to the GMC

$$\boldsymbol{x} = \boldsymbol{x}(\tau) = \boldsymbol{x}(\boldsymbol{x}_0, \tau). \tag{6}$$

The change of the vector potential is related to the change of the coordinates by the relations

$$-\mathrm{grad}u + \dot{\boldsymbol{A}} + \dot{\boldsymbol{x}} \times \boldsymbol{H} = 0, \tag{7}$$

and u satisfies the relation

$$\boldsymbol{H} \cdot \mathrm{grad}u = \boldsymbol{H} \cdot \dot{\boldsymbol{A}}. \tag{8}$$

The dot denotes the derivative with respect to τ.

It can be shown that if the transformation is chosen as

$$\dot{\boldsymbol{x}} = \sigma\boldsymbol{H} - \frac{1}{H^\zeta}\boldsymbol{e}_3 \times \left[\tilde{A} + \mathrm{grad}\nu\right], \tag{9}$$

with ν satisfying equation

$$(\boldsymbol{H}\cdot\nabla)^2\nu - (\boldsymbol{H}\cdot\nabla)(\boldsymbol{H}\cdot\tilde{A}) = 0, \tag{10}$$

the coordinates approach to GMC when $\tau \to \infty$. The parameter σ may be chosen by the third requirement for the coordinates[5]. In the simple case we can choose $\sigma = 0$.

3. Numerical Example

In order to check the algorithm given above, a computer code is written. The following magnetic field is used as a test field.

The model magnetic field is ABC (Arnol'd-Beltrami-Childress) magnetic field in the (x, y, z) Cartesian coordinates added constant magnetic field in the direction of z as follows.

$$\begin{aligned} B_x &= b\cos(\lambda y) + c\sin(\lambda z) \\ B_y &= c\cos(\lambda z) + a\sin(\lambda x) \\ B_z &= a\cos(\lambda x) + b\sin(\lambda y) + B_0 \end{aligned} \tag{11}$$

with $\lambda = 2\pi$, ($a = 0.2$, $b = 0.1$, $c = 0.6$). This magnetic field is periodic in the directions of x, y, z. B_0 is added so that $B_z > 0$.

The (x, y, z) is expanded into Fourier series in terms of (ξ, η, ζ) as follows,

$$\begin{aligned} x &= \xi + \sum_{l,m,n,k} \xi_{lmnk} \sin\lambda(l\xi + m\eta + n\zeta + \frac{k}{4}) \\ y &= \eta + \sum_{l,m,n,k} \eta_{lmnk} \sin\lambda(l\xi + m\eta + n\zeta + \frac{k}{4}) \\ z &= \zeta + \sum_{l,m,n,k} \zeta_{lmnk} \sin\lambda(l\xi + m\eta + n\zeta + \frac{k}{4}). \end{aligned} \tag{12}$$

The index k ($k = 0,1$) distinguishes sine from cosine. The scalar function ν is also expanded by Fourier series.

First, we set $B_0 = 1.0$, $-1 \le l, m, n \le 1$. Figure 1 shows the GMC mesh of $\xi, \eta =$ const. at equal intervals on the $z = 0$, 0.25, 0.5, 0.75 plane in the Cartesian coordinates. The Poincaré map of magnetic surfaces is also overlapped in Fig.1. Here $\zeta = z$.

The deformation of the coordinate mesh follows that of the magnetic surface; for example, the O-point locates at the same point of (ξ, η) regardless of ζ. It is easily seen even with the small number of the Fourier mode, because the variation of the magnetic field is small in this case.

Next, the constant B_0 is lowered to $B_0 = 0.5$, so that the variation of the field in z direction is increased. The agreement between the coordinate mesh and the magnetic surface is not good with the Fourier mode $-1 \le l, m, n \le 1$. When the number of Fourier mode is increased to include the mode $-2 \le l, m, n \le 2$, the situation is improved, as shown in Figure 2.

The convergence of $\tilde{H}$ is shown in Figure 3. The error of $\tilde{H}$ remains finite even if the calculation is iterated more than 10. It should be noted that the error of $\tilde{H}^\xi$, $\tilde{H}^\eta$ naturally contains the part from the place where nested magnetic surface does not exist. On the other hand, the error of $\tilde{H}^\zeta$ is the finite truncation error for the most part. Because the error of $\tilde{H}^\xi$, $\tilde{H}^\eta$ is not very different from the error of $\tilde{H}^\zeta$, the errors of $\tilde{H}$ are caused by the smallness of number of Fourier mode than the breaking of magnetic surfaces. The number of Fourier mode is very important in view of accuracy of coordinates especially in the complicated magnetic field

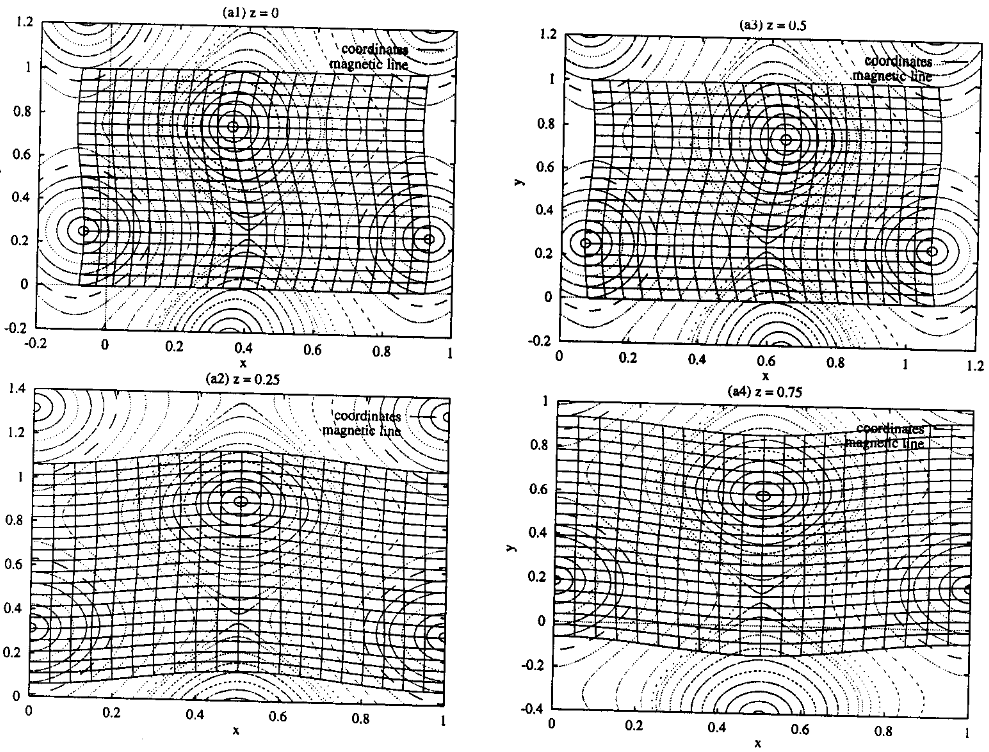

Fig. 1 Poincaré map and Generalized Magnetic Coordinates (B_0 = 1.0, Fourier mode from −1 to 1).
(a1) z = 0 (a2) z = 0.25 (a3) z = 0.5 (a4) z = 0.75

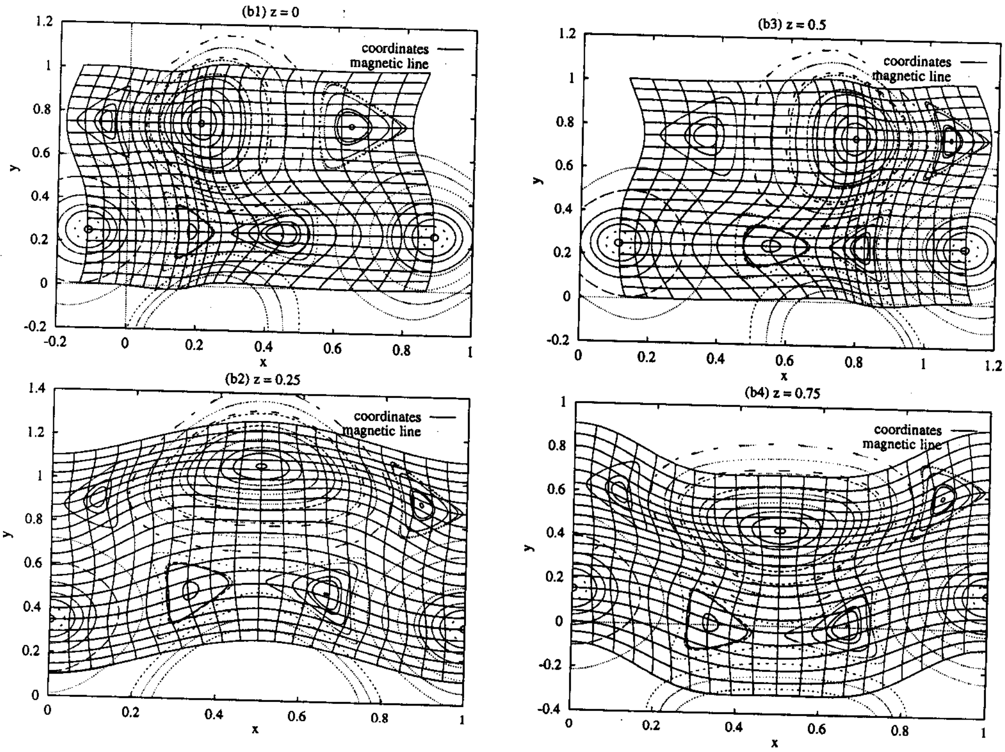

Fig. 2 Poincaré map and Generalized Magnetic Coordinates (B_0 = 0.5, Fourier mode from −2 to 2).
(b1) z = 0 (b2) z = 0.25 (b3) z = 0.5 (b4) z = 0.75

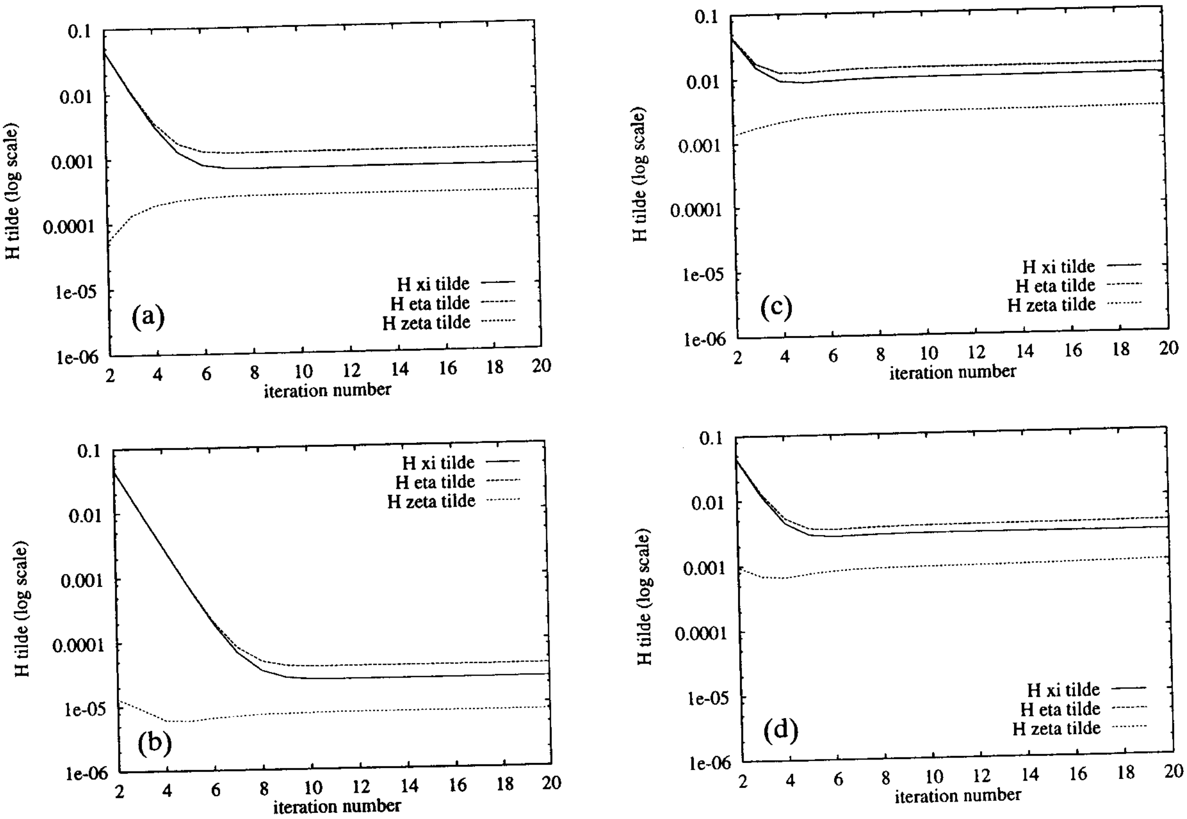

Fig. 3 Convergence of $\tilde{H}$ (Logarithm of square norm of $\tilde{H}$ *vs.* iteration number).
(a) B_0 = 1.0, Fourier mode from −1 to 1.
(b) B_0 = 1.0, Fourier mode from −2 to 2.
(c) B_0 = 0.5, Fourier mode from −1 to 1.
(d) B_0 = 0.5, Fourier mode from −2 to 2.

such as the case of $B_0 = 0.5$.

4. Conclusion

The general method to construct a GMC is applied to the simple magnetic field expressed in an analytic form. When the variation of the magnetic field is small, the good magnetic coordinates are obtained even in the small number of Fourier mode. The poor convergence in the presented result is reduced to the use of the small number of the Fourier mode. We can expect that the situations will be improved by using the sufficient number of Fourier mode.

In the general magnetic configuration of interest the periodic condition in three dimension cannot be used. The region of the existence of the GMC is also limited, which is not known before the calculation. The improvement of method is required in order to treat the general case, but it is left to the future task.

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