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Analytical Theory of Flux Coordinates for Stellarators

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Abstract

Effective method is proposed to construct the flux coordinates for toroidal plasma equilibrium configurations. This method is applied to conventional stellarators. Three kinds of flux coordinates are considered: coordinates with a fixed toroidal angle, Hamada and Boozer coordinates.

Keywords:
plasma equilibrium, flux/magnetic coordinates, conventional stellarator

1. Introduction

Flux (or magnetic) coordinates are defined as curvilinear coordinates (a, x^2, x^3) with a = *const* describing toroidal magnetic surfaces, and x^2 and x^3 as the angular coordinates on these surfaces. For general theory see [1, 2].

Three special kinds of flux coordinates became selected in a practice: coordinates with a fixed toroidal angle [3, 4], Hamada [5] and Boozer [6] coordinates. Although the general theory explains [1] how to transform one system into another, many questions arise in applications [7].

Besides computational aspects [7], there is a great problem of using results expressed in different coordinates. One needs to know, at least, how much one coordinate system differs from the other. Their comparison is difficult because they are introduced in a different way, which makes them isolated.

Recently a unifying approach was proposed [8] allowing to treat and compare different systems easily. Here it is applied to conventional stellarators, which are the systems with planar circular axis and helical magnetic field.

2. General Equations

Magnetic field $\boldsymbol{B}$ can be represented in an arbitrary flux coordinates as (see, for example, [1])

$$2\pi \boldsymbol{B} = \nabla\Psi \times \nabla x^3 + \nabla\Phi \times \nabla x^2 + \nabla a \times \nabla\eta, \quad (1)$$

where Ψ and Φ are the poloidal and toroidal fluxes respectively, Fig. 1, and η is the double-periodic function depending on the choice of x^2 and x^3.

The freedom in selecting x^2 and x^3 can be used to simplify the expression (1) for the magnetic field. For example, to make field lines straight by $x^2 \to x^2 + \eta/\Phi'$, where prime means the derivative with respect to a. Formally, it corresponds to $\eta = 0$ in Eq. (1). If we assume after that $x^3 = \zeta$, where ζ is the given geometrical coordinate (such as the usual toroidal angle), no more

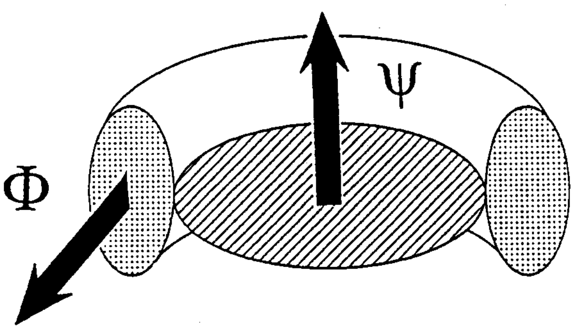

Fig. 1 Toroidal and poloidal magnetic fluxes

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freedom remains. Otherwise, in addition to $\eta = 0$ it is possible to put one more constraint. For Hamada and Boozer coordinates it is imposed as a requirement for a functional dependence of the Jacobian

$$\sqrt{g} = ([\nabla a \times \nabla x^2] \cdot \nabla x^3)^{-1}. \qquad (2)$$

Finally, coordinates (a, θ_g, ζ) with a given x^3 (toroidal angle), which can be called "basic" coordinates, are defined by

$$\eta = 0, \qquad x^3 = \zeta. \qquad (3)$$

Hamada coordinates (a, θ_H, ζ_H) are introduced by

$$\eta = 0, \qquad \sqrt{g_H} = \frac{V'}{4\pi^2}, \qquad (4)$$

and Boozer coordinates (a, θ_B, ζ_B) by

$$\eta = 0, \qquad \sqrt{g_B} = \frac{V'}{4\pi^2}\frac{\langle \boldsymbol{B}^2 \rangle}{\boldsymbol{B}^2}. \qquad (5)$$

Brackets $\langle ... \rangle$ denote the usual volume averaging between neighboring flux surfaces:

$$\langle X \rangle \equiv \frac{d}{dV}\int_V X d^3 \boldsymbol{r} = \frac{1}{V'}\int X\sqrt{g}\, dx^2\, dx^3. \qquad (6)$$

Definitions (3)—(5) are mathematically perfect, but not informative. Except $x^3 = \zeta$, they show some secondary properties, but there is no any hint on the x^2 and x^3. Also, it is not clear how to compare two systems even in the case of similar definitions (4) and (5).

The alternative way proposed recently [8] is based on the relation

$$\frac{1}{\sqrt{g}} = \frac{4\pi^2}{V'}\frac{f}{\langle f \rangle} \qquad (7)$$

and two equations

$$(2\pi \boldsymbol{B} - \nabla a \times \nabla \eta) \cdot \nabla x^2 = -4\pi^2 \frac{\Psi'}{V'}\frac{f}{\langle f \rangle}, \qquad (8)$$

and

$$(2\pi \boldsymbol{B} - \nabla a \times \nabla \eta) \cdot \nabla x^3 = 4\pi^2 \frac{\Phi'}{V'}\frac{f}{\langle f \rangle}. \qquad (9)$$

Relation (7) is the direct consequence of definition (6), and (8) and (9) are two components of the vector equality (1).

There are four functions: x^2, x^3, η and f. Any two of them can be considered as "free parameters". If η and f are given, Eqs. (8) and (9) turn into equations for x^2 and x^3. Traditionally, Hamada and Boozer coordinates are introduced in such a way, see Eqs. (4) and (5). We propose to use the algorithm for *any* coordinate system. Arbitrary flux coordinates can be considered in the frame of a single unified approach because their Jacobians $\sqrt{g}$ are represented now as the elements of one-parameter family in Eq. (7). It allows to make a "smooth transition" from one system to another.

The most important is that straight field line (SFL) coordinates (a, θ_g, ζ) with a fixed toroidal angle can be prescribed in the same manner as Hamada and Boozer coordinates. In addition to Eq. (3), they can be introduced by Eqs. (7)—(9) with [8]

$$\eta = 0, \qquad f = \boldsymbol{B} \cdot \nabla \zeta. \qquad (10)$$

Hamada coordinates correspond to

$$\eta = 0, \qquad f = 1, \qquad (11)$$

and Boozer coordinates to

$$\eta = 0, \qquad f = \boldsymbol{B}^2. \qquad (12)$$

These relations are more convenient for coordinate comparison than traditional Eqs. (3)—(5).

Let us note that "basic" coordinates are, in fact, introduced here by three functions: η, f and ζ, although initially there are only two degrees of freedom. This important nontrivial trick, allowing to include the "basic" coordinates into the unified approach, becomes possible because Eq. (9) turns into identity at the choice of Eq. (10).

3. Comparison of Coordinates

In toroidal systems magnetic field is inhomogeneous. It is the only reason why "basic", Hamada and Boozer coordinates are different. In conventional stellarators with planar circular axis

$$\boldsymbol{B} = \bar{\boldsymbol{B}} + \tilde{\boldsymbol{B}}. \qquad (13)$$

Here $\tilde{\boldsymbol{B}}$ is the helical field, and $\bar{\boldsymbol{B}}$ is the axially symmetric field,

$$\bar{\boldsymbol{B}} = B_t \boldsymbol{e}_\zeta + \boldsymbol{B}_p, \qquad (14)$$

B_t is the toroidal component of $\bar{\boldsymbol{B}}$, and the subscript p denotes the poloidal component, which is usually small. Representations (13) and (14) help to show explicitly the origin and degree of the difference.

An attractive feature of Eqs. (7)—(9) is that normalization or even dimension of the function f can be arbitrary. It allows to express these functions in Eqs. (10)—(12) in the same units and in the similar form to make easier their comparison.

It is clear that $f = 1$ in Eq. (11) can be replaced by $f = B_0^2$, where B_0 is the toroidal field at geometrical axis $r = R$. And proper coefficient for $\boldsymbol{B} \cdot \nabla \zeta$ in Eq. (10) could be RB_0. At low β, which is typical for conventional stellarators, diamagnetic contribution to rB_t is very small, and $rB_t \cong RB_0$ with a high accuracy. Then we get

$$\boldsymbol{B}\cdot\nabla\theta_g = -2\pi\frac{d\Psi}{dV}\frac{B_t^2 + B_t\widetilde{B}_\zeta}{\langle B_t^2 + B_t\widetilde{B}_\zeta\rangle} \tag{15}$$

for "basic" coordinate system,

$$\boldsymbol{B}\cdot\nabla\theta_B = -2\pi\frac{d\Psi}{dV}\frac{\boldsymbol{B}^2}{\langle \boldsymbol{B}^2\rangle} \tag{16}$$

for Boozer coordinates, and

$$\boldsymbol{B}\cdot\nabla\theta_H = -2\pi\frac{d\Psi}{dV}\frac{B_0^2}{\langle B_0^2\rangle}. \tag{17}$$

for Hamada coordinates.

By definition, any SFL coordinates must satisfy

$$\boldsymbol{B}\cdot\nabla(x^3 - qx^2) = 0, \tag{18}$$

where $q = -\Phi'/\Psi'$. Therefore, corresponding equations for toroidal angles are similar to Eqs. (15)–(17).

Equations (15)–(17) show that "basic" and Boozer coordinates are very close to each other, but Hamada coordinates stay apart. Such result was obtained in Ref. [7] numerically for some particular choice of parameters. In our case we come to the conclusion even without solving any equation. It must be valid in a general case, because always $B_t^2 + B_t\widetilde{B}_\zeta$ is much better approximation of $\boldsymbol{B}^2$ than B_0^2.

4. Surface Functions

Equations for "basic" and Boozer coordinates contain surface functions $\langle\boldsymbol{B}\cdot\nabla\zeta\rangle$ and $\langle\boldsymbol{B}^2\rangle$. By definition (6), they must be calculated through the integrals over the volume V inside the magnetic surface. For conventional stellarator the integration over ζ can be done analytically with a result [9]

$$\langle X\rangle = \langle X + \operatorname{div}(X\delta\boldsymbol{r})\rangle_0, \tag{19}$$

where $\langle\ldots\rangle_0$ stays for 2D "quasi-tokamak" averaging. Then

$$\langle\widetilde{X}\rangle = -\langle\widetilde{\boldsymbol{B}}\cdot\nabla(\frac{r}{B_t}\hat{\widetilde{X}})\rangle_0, \tag{20}$$

where $\hat{\widetilde{X}}$ is the oscillating part of the integral $\int\widetilde{X}\,d\zeta$. It allows to get

$$\langle B_t\widetilde{B}_\zeta + \widetilde{\boldsymbol{B}}^2\rangle = 0. \tag{21}$$

With a help of this useful relation we obtain immediately

$$\langle\boldsymbol{B}^2\rangle = \langle B_t^2 + \boldsymbol{B}_p^2 - \widetilde{\boldsymbol{B}}^2\rangle$$
$$= \frac{r^2B_t^2}{R^2}\left\{1 - \langle\Omega - \boldsymbol{B}_p^2/B_0^2\rangle\right\}, \tag{22}$$

where

$$\Omega \equiv 1 - \frac{R^2}{r^2} + \frac{\langle\widetilde{\boldsymbol{B}}^2\rangle_\zeta}{B_0^2}. \tag{23}$$

It was taken into account that rB_t = const. on the averaged magnetic surface [1].

This property together with Eq. (21) allows to write

$$\langle\boldsymbol{B}^2\rangle = rB_t\langle\boldsymbol{B}\cdot\nabla\zeta\rangle + \langle\boldsymbol{B}_p^2\rangle, \tag{24}$$

which relates $\langle\boldsymbol{B}\cdot\nabla\zeta\rangle$ with $\langle\boldsymbol{B}^2\rangle$.

Poloidal field is small in stellarators and can be disregarded in Eqs. (22) and (24). Then $\langle\boldsymbol{B}\cdot\nabla\zeta\rangle$ and $\langle\boldsymbol{B}^2\rangle$ are determined by rB_t and Ω. The first value is almost constant all over the plasma cross-section (diamagnetic change of rB_t is very small). And $\langle\Omega\rangle$ is always small. Then there is no need to distinguish $\langle\boldsymbol{B}\cdot\nabla\zeta\rangle$ and $\langle\boldsymbol{B}^2\rangle$ from $\langle\boldsymbol{B}\cdot\nabla\zeta\rangle_0$ and $\langle\boldsymbol{B}^2\rangle_0$ in solving Eqs. (8), (9) or (15), (16).

5. Magnetic Differential Equation for a Stellarator

To find x^2 and x^3, one needs to solve Eqs. (8) and (9) with given η and f. For basic, Hamada and Boozer coordinates ($\eta = 0$) they turn into magnetic differential equations

$$\boldsymbol{B}\cdot\nabla x = y. \tag{25}$$

where $\boldsymbol{B}\cdot\nabla\zeta$, $\boldsymbol{B}^2$ and their averaged $\langle\boldsymbol{B}\cdot\nabla\zeta\rangle$, $\langle\boldsymbol{B}^2\rangle$ are present. These values and operator $\boldsymbol{B}\cdot\nabla$ contain the magnetic field $\boldsymbol{B}$, which is given by Eq. (13) for conventional stellarators. Condition $|\widetilde{\boldsymbol{B}}| \ll |\overline{\boldsymbol{B}}|$ allows to use so-called stellarator expansion for solving (25).

The method was proposed more than 35 years ago [10] for a large aspect ratio systems, but it can be modified to avoid this restriction [1, 9]. To apply the technique described in Ref. [9], we have to represent all values like Eq. (13) and to split Eq. (25) on two parts:

$$\overline{\boldsymbol{B}}\cdot\nabla\bar{x} + \langle\widetilde{\boldsymbol{B}}\cdot\nabla\widetilde{x}\rangle_\zeta = \bar{y}, \tag{26}$$

$$\overline{\boldsymbol{B}}\cdot\nabla\widetilde{x} + \widetilde{\boldsymbol{B}}\cdot\nabla\bar{x} = \widetilde{y}. \tag{27}$$

Here $\langle X\rangle_\zeta$ and $\overline{X}$ mean the same: axisymmetric part of function X.

The last equation, which was linearized, gives

$$\widetilde{x} = -\delta\boldsymbol{r}\cdot\nabla\bar{x} + \frac{r}{B_t}\hat{\widetilde{y}}, \tag{28}$$

where

$$\delta t \equiv \frac{r}{B_t}\hat{\widetilde{\boldsymbol{B}}}_p. \tag{29}$$

With $\tilde{x}$ expressed as Eq. (28), Eq. (26) after integration turns into

$$(\bar{\boldsymbol{B}} + \boldsymbol{B}^*) \cdot \nabla \bar{x} = \bar{y} - \langle \tilde{\boldsymbol{B}} \cdot \nabla (\frac{r}{B_t} \hat{\tilde{y}}) \rangle_\zeta, \tag{30}$$

where

$$\boldsymbol{B}^* \equiv \frac{1}{2\pi} [\nabla \psi_v \times \nabla \zeta] \tag{31}$$

can be called the "effective poloidal field", and ψ_v is the poloidal flux of the helical field:

$$\psi_v \equiv \frac{\pi r^2}{B_t} \langle [\tilde{\boldsymbol{B}} \times \hat{\tilde{\boldsymbol{B}}}] \cdot \boldsymbol{e}_\zeta \rangle_\zeta$$

$$= \frac{2\pi r^2}{B_t} \langle \tilde{B}_z \hat{\tilde{B}}_r \rangle_\zeta . \tag{32}$$

Identity

$$\langle \tilde{\boldsymbol{B}} \cdot \nabla (\delta \boldsymbol{r} \cdot \nabla \bar{y}) \rangle_\zeta = - \boldsymbol{B}^* \cdot \nabla \bar{y} \tag{33}$$

must be used to get Eq. (30) from Eq. (26).

As a result, the solution of the Eq. (25) looks like

$$x = \bar{x} + \tilde{x} = \bar{x} - \delta \boldsymbol{r} \cdot \nabla \bar{x} + \frac{r}{B_t} \hat{\tilde{y}} , \tag{34}$$

and $\bar{x}$ must be found from the two-dimensional equation (30), which is much simpler than Eq. (25).

6. General Analytical Solutions for a Stellarator

From Eqs. (15)–(17) the easiest is the Eq. (17) for Hamada angle θ_H. It corresponds to MDE (25) with $y = 1$ and $2\pi x_H = - V'(\Psi)\theta_H$. It follows from Eqs. (28) and (30) that in this case $\tilde{x}_H = - \delta \boldsymbol{r} \cdot \nabla \bar{x}_H$, and

$$\bar{x}_H = \int \frac{\mathrm{dl}}{(\boldsymbol{B}_p + \boldsymbol{B}^*) \cdot \boldsymbol{p}}, \tag{35}$$

where integration is performed along the transverse cross-section of the averaged magnetic surface, $\boldsymbol{p}$ is the unit vector tangent to this contour.

For basic coordinates $\tilde{y} \neq 0$, but it makes rather small contribution into the oscillating part of Eq. (34). Then

$$\theta_g = \theta - \delta \boldsymbol{r} \cdot \nabla \theta \tag{36}$$

with

$$\theta = \frac{1}{q} \int \frac{B_t}{r} \frac{\mathrm{dl}}{(\boldsymbol{B}_p + \boldsymbol{B}^*) \cdot \boldsymbol{p}} . \tag{37}$$

It is easy to get similar expressions for Boozer coordinates, which are very close to basic coordinates.

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