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Plasma Equilibrium and Rotation in Stellarators

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Abstract

In a dissipative model of plasma equilibria in stellarators the viscous forces and inertial forces are added to the force balance of ideal MHD. This model allows one to describe a rotating equilibrium which is of interest in the theory of H-mode confinement. Existence of solutions can be proven using the methods of viscous hydrodynamics. Uniqueness only occurs at small Hartmann and Lundquist numbers. The viscous damping is evaluated numerically showing that optimised stellarators of the Helias type exhibit smaller poloidal damping than standard stellarators. Island formation in the boundary region may inhibit the onset of rotation since enhanced viscous damping or magnetic pumping arises in the neighbourhood of islands.

Keywords:

stellarator, dissipative equilibrium, rotation, viscous damping

The standard method in computing 3-dimensional stellarator equilibria is to solve the force balance $\boldsymbol{j}\times\boldsymbol{B}=\nabla p$ either by looking for an extremum of the energy integral or by applying Spitzer's iterative scheme $\boldsymbol{j}_{n+1}\times\boldsymbol{B}_{n}=\nabla p$; $\nabla\times\boldsymbol{B}_{n+1}=\mu_0\boldsymbol{j}_{n+1}$, $n=0,1,2...$. However, there exists no proof of convergence and since the parallel plasma current tends to develop singularities [1] it is to be suspected that the procedure does not converge in general. From Ohm's law with finite resistivity it follows that also the electric potential becomes singular on rational magnetic surfaces which leads to an infinite shear flow of the $\boldsymbol{E}\times\boldsymbol{B}$ velocity. These singularities will be cut off by finite plasma viscosity. Furthermore, rotating plasma equilibria, which are of interest in the context of H-mode confinement cannot be computed in the ideal MHD-model. The Navier-Stokes model of plasma equilibria

$$\nabla\cdot(\rho\boldsymbol{v}{:}\boldsymbol{v}+\pi(\boldsymbol{v}))=-\nabla p+\boldsymbol{j}\times\boldsymbol{B} \qquad (1)$$

together with Ohm's law $-\nabla_{\Phi}+\boldsymbol{v}\times\boldsymbol{B}=\eta\boldsymbol{j}$ and $\nabla\cdot(\rho\,\boldsymbol{v})=S(\boldsymbol{x})$, $p=p(\rho,T)$ removes these difficulties. $\pi(\boldsymbol{v})$ is the Braginskii viscous tensor, which is linear in the derivatives of the velocity $\boldsymbol{v}$. An alternative to the equation of state is the adiabatic law $\boldsymbol{v}\cdot\nabla(\ln(p/\rho^{\gamma}))=0$.

The iterative scheme $\boldsymbol{B}_n \rightarrow (\boldsymbol{j}_{n+1},\boldsymbol{v}_{n+1}) \rightarrow \boldsymbol{B}_{n+1}$ starts with a given magnetic field and computes the pair $(\boldsymbol{j}_{n+1},\boldsymbol{v}_{n+1})$ and a new magnetic field $\boldsymbol{B}_{n+1}$. The convergence is equivalent to the existence of a fixed point in a Banach space of vector fields $(\boldsymbol{j},\boldsymbol{v})$. The method is well established in the theory of viscous hydrodynamics [2, 3] and has been applied to magnetohydrodynamic flow by Gunzberger *et al.* [4] and Spada and Wobig [5]. In this model there are two dimensionless parameters, the Hartmann number $H=BL(\sigma/\eta)^{1/2}$ and the Lundquist number $R_{\mathrm{m}}=\sigma uL$ (σ = conductivity, η = viscosity, u = reference velocity, L = length scale) and it can be shown that any solution is unique if the Lundquist number and the Hartmann number are small enough. At large values of H and R_{m} bifurcations and multiple solutions may occur. This is similar to hydrodynamic flow where bifurcation occurs at large Reynolds numbers. The afore-mentioned theories are still incomplete since plasma density and temperature are not

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computed self-consistently. With given density and convective velocity the heat conduction equation yields a temperature T and using the equation of state, $\rho=\rho(p,T)$, this procedure provides us with a new density. In this extended model the equilibrium is a fixed point of the map $(v_n, j_n, \rho_n) \rightarrow (v_{n+1}, j_{n+1}, \rho_{n+1})$. In the theory of gas dynamics the mathematical analysis of compressible flow is given in ref. [6]. The heat conduction equation introduces another reason for bifurcations which is due to the non-linear radiation function $L_z(\mathrm{T})$.

A necessary condition for the existence of rotating equilibria is a finite particle source $S(x) \neq 0$ and a radial plasma diffusion. Without a particle source ($S=0$) and zero velocity $v_n=0$ normal to the plasma boundary the work done by viscous forces and the Ohmic dissipation is zero which implies $v \equiv 0$ and $j \equiv 0$. The dissipated energy of the system is

$$W[v,j] = \int_\Omega \left(v \cdot \nabla \cdot \pi(v) + \eta j^2 \right) \mathrm{d}^3 x \tag{2}$$

From the force balance, Ohm's law and the boundary conditions we obtain $\mathrm{W}[v, j]=0$, which yields the result above. In order to get this result use of the full Braginskii viscous tensor including shear viscosity has to be made. Only in this case the viscous operator $\nabla \cdot \pi(v)$ is positive definite.

Given a rotating equilibrium, a surface-averaged force balance can be established which elucidates the various spin-up mechanisms — Stringer spin-up and turbulent Reynolds stresses — and balances these driving forces against the viscous damping forces. The complicated structure of general stellarator equilibria is overcome by using the Hamada coordinate system [7]. In the one-fluid model, averaging the force balance over a pressure surface yields the two equations

$$\begin{aligned} &< \boldsymbol{j} \cdot \nabla \cdot (\rho v{:}v + \pi(v)) > \; = 0, \\ &< \boldsymbol{B} \cdot \nabla \cdot (\rho v{:}v + \pi(v)) > \; = 0, \end{aligned} \tag{3}$$

Viscous damping parallel to the plasma current and the magnetic field lines is balanced by surface averaged inertial forces. Since viscous forces and inertial forces in rotating equilibria are small compared to the pressure gradient and the Lorentz forces we have $\boldsymbol{B} \cdot \nabla \boldsymbol{p} \approx 0$, magnetic surfaces and pressure surface nearly coincide. Using this property we may approximate the magnetic field by the field of an ideal equilibrium and introduce the Hamada coordinate system to evaluate the surface averaged viscous forces. In terms of the poloidal base vector e_p the viscous damping forces are

$$\begin{aligned} &< e_\mathrm{p} \cdot \nabla \cdot \pi > \; = \; <(p_\parallel - p_\perp) e_\mathrm{p} \cdot \frac{\Delta B}{B}> \\ &< B \cdot \nabla \cdot \pi > \; = \; <(p_\parallel - p_\perp) B \cdot \frac{\Delta B}{B}> \end{aligned} \tag{3}$$

This formulation holds for all regimes of collisionality; in the collisionless regime the anisotropy of the pressure must be computed by neoclassical theory, in the collisional regime it can be found from the Braginskii viscosity. In this regime it is identical with the magnetic pumping effect.

$$\begin{pmatrix} -< e_\mathrm{p} \cdot \nabla \cdot \pi > \\ < B \cdot \nabla \cdot \pi > \end{pmatrix} = 3 \tau P \begin{pmatrix} C_\mathrm{p} & C_\mathrm{b} \\ C_\mathrm{b} & C_\mathrm{t} \end{pmatrix} \begin{pmatrix} E \\ \Lambda \end{pmatrix} \tag{4}$$

The coefficients are

$$C_\mathrm{p} = <\left(e_\mathrm{p} \cdot \frac{\nabla B}{B} \right)^2>; \; C_\mathrm{t} = <\left(B \cdot \frac{\nabla B}{B} \right)^2>$$

$$C_\mathrm{b} = <\left(e_\mathrm{p} \cdot \frac{\nabla B}{B} \right)\left(B \cdot \frac{\nabla B}{B} \right)> \tag{5}$$

τ is the ion-ion collision time. The plasma velocity is in lowest order given by

$$v_\mathrm{o} = -E(\psi) e_\mathrm{p} + \Lambda(\psi) B \tag{6}$$

Viscous damping of rotation depends on the details of the magnetic field and numerical evaluation shows significant differences in various stellarator configurations [8]. These geometrical coefficients are large in standard stellarators of the l=2-type, however they are appreciably smaller in optimized stellarators of the Helias type. Therefore rotating equilibria may be more easily obtained in optimized stellarators than in conventional stellarators.

It is shown that the presence of magnetic islands enhances viscous damping up to a factor of two in the neighbourhood of these islands. Magnetic islands have been observed in the Wendelstein 7-AS experiment and it has been shown how the attainment of H-mode confinement in this experiment is related to the absence of magnetic islands [9].

Poloidal rotation with shear flow is one of the key elements in the theory of H-mode confinement in toroidal systems. There are various driving forces which may excite poloidal and toroidal rotation, Stringer spin-up,

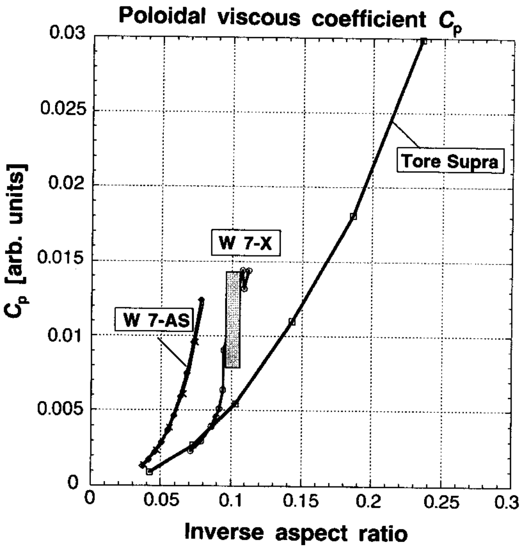

Fig. 1. Poloidal viscous coefficient C_p in Wendelstein 7-AS, Wendelstein 7-X and an axisymmetric device (Tore Supra). In W7-X, poloidal damping is reduced to the level of an axisymmetric configuration, only close to the boundary does the presence of magnetic islands tend to increase poloidal viscous damping.

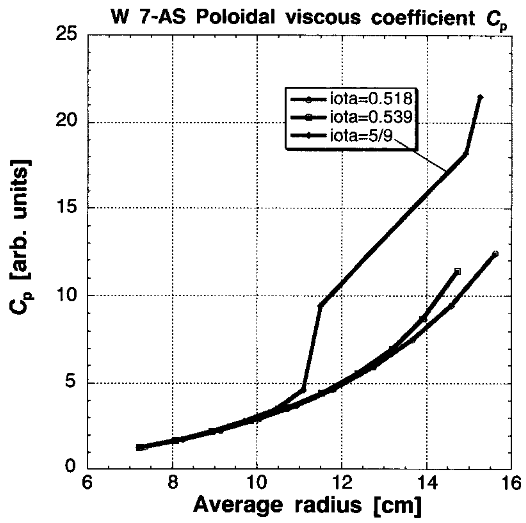

Fig. 2. Poloidal viscous coefficient in W7-AS in the regime $\iota > 0.5$. The upper curve shows the effect of the $\iota = 5/9$ island. Viscous damping in the neighbourhood of the island is enhanced by a factor of two. Reducing the rotational transform slightly, shifts the islands beyond the last closed surface. In the two lower curves the plasma is affected neither by islands nor by contact with a material limiter. This is the region where in W7-AS H-mode-like confinement has been observed.

turbulent Reynolds stresses and lost orbits. In Refs. [7] and [8], the surface-averaged inertial forces have been analysed explicitely showing the specific role of the radial diffusion losses. It is the Coriolis effect which couples diffusive velocity into poloidal rotation. The dominating term in the Coriolis forces is the Stringer spin-up, either driven by classical and Pfirsch-Schlüter diffusion or by anomalous diffusion. In case of a turbulent plasma the inertial forces in Eq. (2) are to be modified by additional turbulent Reynolds stresses which especially in the initial phase of rotation may play a decisive role.

The results described above only consider the linear phase of plasma rotation — assuming that such an equilibrium exists. In the fully developed rotating state the shear flow reduces the anomalous losses and thus the inertial driving term. Viscous damping in the plateau and long-mean-free-path-regime is also reduced by plasma rotation. However, the present linear analysis is useful for comparing the influence of magnetic geometry on viscous damping. Reducing the poloidal and toroidal variation of B on magnetic surfaces reduces the magnetic pumping effect (or viscous damping) which balances the classical or anomalous driving terms. It has been shown that optimization of stellarator configuration with respect to Pfirsch-Schlüter currents and neo-classical diffusion also reduces poloidal viscous damping to the level of axisymmetric configurations. Therefore in Wendelstein 7-X and related configurations a significant H-mode effect may be expected.

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