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Spatial-Axis Stellarators

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Abstract

The short review of the theoretical results obtained in Kurchatov Institute in the field of helical magnetic confinement systems and which could be useful at present are given. Some arguments are adduced in favor of the Heliac-type quasisymmetrical systems. The result based on 3-D codes VMEC and TERPSICHORE is presented, demonstrating rather high ballooning stable $\langle\beta\rangle \sim 3\%$ in the four-period slightly corrugated heliac-type system. The way to avoid the locally trapped particles which are responsible for enhanced transport in stellarators, less restrictive than the condition of quasisymmetry is discussed on the base of "normal" systems having topology of lines B = const. on the magnetic surfaces like at axial, helical or poloidal symmetry, *i.e.* without local extremes of B on magnetic surfaces.

Keywords:

helical system, flux coordinates, plasma stability, quasisymmetry, normal systems

1. Introduction

At present we observe a revival of the helical plasma confinement systems which are directly connected with L. Spitzer invitation of the space-figure-eight stellarator [1]. In Kurchatov Institute, the theory of such systems starts in the middle of sixties in connection with construction of small figure-eight stellarator consisting of four semi-tori. For calculation of the plasma shift, the Mercier quasicylindrical orthogonal coordinate system was used [2]. The Mercier quasiflute instability was studied in Ref. [3] on a base of curvilinear flux coordinates with both contravariant and covariant representations of the magnetic field $\boldsymbol{B}$:

$$2\pi B^{i} = \{0, -\Psi'/\sqrt{g}, \Phi'/\sqrt{g}\},$$

$$2\pi B_{i} = \{-\nu + \partial\varphi/\partial\alpha, J + \partial\varphi/\partial\theta, F + \partial\varphi/\partial\zeta\}.$$

In vector form: $\quad 2\pi\boldsymbol{B} = \nabla\Psi\times\nabla\zeta + \nabla\Phi\times\nabla\theta,$

and $\quad 2\pi\boldsymbol{B} = J\nabla\theta + F\nabla\zeta - \nu\nabla\alpha + \nabla\varphi,$

(that corresponds to $\quad 2\pi\boldsymbol{A} = \Phi\nabla\theta + \Psi\nabla\zeta,$ $\boldsymbol{B} = \nabla\times\boldsymbol{A}$).

Here $\sqrt{g} = (\nabla\theta\times\nabla\zeta\cdot\nabla a)^{-1}$, $J(a)$, $\Phi(a)$ are toroidal current and magnetic flux and $F(a)$, $\Psi(a)$, are poloidal external current and magnetic flux, a is a label of the toroidal magnetic surface, *e.g.* the normalized radius $a(r)$ = const, or the value $V(a)$ inside the magnetic surface, or $\chi(a) = \Psi_{\text{axe}} - \Psi(a)$ etc. Functions ν and φ are necessary to satisfy conditions $\boldsymbol{B}\cdot\nabla a = 0$ and $\nabla\cdot\boldsymbol{B} = 0$. By definition, the rotational transform is $\iota = (\boldsymbol{B}\cdot\nabla\theta)/(\boldsymbol{B}\cdot\nabla\zeta) = \chi'(a)/\Phi'(a) = -\Psi'(a)/\Phi'(a)$.

One of useful conclusion of those old researches was discovering a possibility of the local sufficient stability criterion for the ideal MHD modes in the case when there is no net toroidal current [4]. This criterion is obtained from simple consideration of the potential energy of the ideal incompressible displacements and has a form [5]:

$$\frac{(\boldsymbol{j}\cdot\boldsymbol{B})^2}{B^2|\nabla a|^2} - \frac{\langle \boldsymbol{j}\cdot\boldsymbol{B}\rangle}{B^2}\Lambda + \frac{p'}{<B^2>}\left[\omega' + p'\left(\frac{\langle B^2\rangle}{B^2} - 1\right)\right] + k^2|\nabla a|^2 < 0.$$

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Here $\Lambda = (\Phi')^2 \iota' / V'$, $\langle \boldsymbol{j} \cdot \boldsymbol{B} \rangle = (J'F - F'J)/V'$. Without the net current $J(a) = 0$, only the magnetic well could provide fulfillment of this condition. Besides it is desirable to have:

- small Pfirsch-Schlüter current (the first term),
- small $k^2 |\nabla a|^2$.

If $J(a) \neq 0$, then the condition $J' \iota' > 0$ is desirable.

A computational information about the all terms of sufficient criterion could be useful for better understanding the way of confinement system optimization.

The less restrictive sufficient stability criterion, proposed by D. Lortz *et al.* [6], takes into account the bending of the magnetic field lines. By neglecting the all explicitly positive terms in δW except the first and the last ones, one obtains from functional $\delta W(\xi)$ the next Lagrangian equation:

$$\boldsymbol{B} \cdot \nabla (\boldsymbol{B} \cdot \nabla \xi / |\nabla a|^2) + K\xi = 0.$$

A finite solution corresponds to instability. At present, with existing codes it could be useful to analyze this equation numerically.

2. Quasisymmetric Systems

Our study of the quasisymmetric (QS) toroidal system was inspired by J. Nührenberg and R. Zille's [7] discovering a possibility to satisfy condition of quasisymmetry, which was proposed by A. Boozer [8] as a means to avoid the locally trapped particles due to conservation of the canonical momentum of the drift particle. In this case one will have deal with neoclassical transport like in axisymmetric tokamak. In our first paper concerning the QS systems [9], it was discovered a possibility to have the QS conditions at small number of periods not only in systems of Helias-type but also of the Heliac-type systems. Let recall the difference between these systems.

In Heliac-type systems the magnetic surface cross-section rotates mainly together with the magnetic axis principal normal. Thus, being vertically elongated at the maximum axis curvature, it will be vertically elongated also at the middle of the system period. This allows to conserve the favorable D-shape form for stability along the all period. Because of this we have concentrated on the Heliac-type systems. In Helias-type systems the magnetic surface cross-section rotates twice slowly thus becomes horizontally elongated at the middle of period.

The types of quasisymmetry depend on direction of the lines B = const. on magnetic surface. They can go along the torus ("toroidal" quasisymmetry) or around the magnetic axis ("poloidal" quasisymmetry). Near the magnetic axis $B = B_0/(1 - kx) \approx B_0(1 + k\rho \cos\theta)$, where x is directed along the axis principal normal $\boldsymbol{\nu}$. If in Boozer coordinates the line θ_v = const., a = const. has topology of the line B = const. on magnetic surface, then the QS condition is:

$$\partial B / \partial \zeta_v = 0, \qquad B = B(a, \theta_v),$$

or in invariant form:

$$(F\boldsymbol{B} + \boldsymbol{B} \times \nabla \Psi) \cdot \nabla B = 0.$$

Here $F(a)$ and $\Psi(a)$ are the current and magnetic flux inside the contour θ_v = const., a = const. If the poloidal strip θ_h = const. of the Boozer coordinates (a, θ_h, ζ_h) is counted from the torus "hole", not rotating around the magnetic axis, then

$$\theta_v = \theta_h + N\zeta_h, \qquad B = B(a, \theta_h + N\zeta_h).$$

It is the case of the quasihelical symmetry. It is natural to consider the contours ζ_v = const., ζ_h = const. having the same topology. In this case $J_h = J_v = J$, $\Phi_h = \Phi_v = \Phi$.

From $\boldsymbol{B} \cdot \nabla \theta_v = \boldsymbol{B} \cdot \nabla \theta_h + N\boldsymbol{B} \cdot \nabla \zeta_h$ and from covariant representation of $\boldsymbol{B}$ it follows:

$$\iota_v = \iota_h + N,$$

$$\Psi_v = \Psi_h - N\Phi, \qquad F_v = F_h + NJ.$$

The condition of the poloidal quasisymmetry is

$$\partial B / \partial \theta_v = 0, \qquad B = B(a, \zeta_v)$$

in Boozer coordinates, or

$$(J\boldsymbol{B} + \boldsymbol{B} \times \nabla \Phi) \cdot \nabla B = 0$$

in invariant form. At zero net current, $J = 0$, it is condition of the "isodynamic" system, where the particles drift along the magnetic axis that leads to the classical (!) transport. The systems satisfying this condition at $k = 0$ are known as orthogonal mirror systems. Unfortunately, the term $kx = k\rho \cos\theta$ in expression for B could not be excluded at $k \neq 0$ (in Boozer coordinates).

In our consideration the form of the magnetic axis is prescribed. Usually it is taken as a helical line $\boldsymbol{r} = \boldsymbol{r}_0(s)$ on a reference torus. The radius-vector of the arbitrary point is described by $\boldsymbol{r} = \boldsymbol{r}_0(s) + \rho \boldsymbol{e}_\rho$, where the unit vector $\boldsymbol{e}_\rho$ lies in the plane s = const. which is orthogonal to the axis, thus $\boldsymbol{e}_\rho \cdot \nabla s = 0$. Let ω is the angle between the principal normal $\boldsymbol{\nu}$ and $\boldsymbol{e}_\rho$, then the metric of the quasicylindrical coordinate system $(\rho, \omega, \zeta = 2\pi s/L)$ is represented by the expression (see Ref.[10])

$$dl^2 = d\rho^2 + \rho^2 d\omega^2 + 2\kappa\rho^2 R\, d\omega\, d\zeta + R^2[(1 - \kappa\rho\cos\omega)^2 + \kappa^2\rho^2]\, d\zeta^2.$$

Let the near-axis elliptical magnetic surface cross-section rotates according to the relation $\theta_e = \omega + \delta(\zeta) = \omega + n\zeta + \tilde{\delta}(\zeta)$, where θ_e is counted from the top of the ellipse. Following C. Mercier [11] we denote its semiaxis by $a\exp(-\eta/2)$ and $a\exp(\eta/2)$. Thus the equation of the ellipse in parametric form is

$$p\cos\theta_e = a\exp(-\eta/2)\cos\theta_e^*$$
$$p\sin\theta_e = a\exp(\eta/2)\sin\theta_e^*.$$

As a result we obtain for the linear in ka term in expression for $B \approx B_0(1+kx)$:

$$\begin{aligned} kx &= k\rho\cos(\theta_e - \delta) \\ &= ka\{\exp(-\eta/2)\cos\delta\cos\theta_e^* \\ &\qquad + \exp(\eta/2)\sin\delta\sin\theta_e^*\}. \end{aligned}$$

Returning to the poloidal variable θ_v^* counted from the principal normal (thus QS condition to be $B = B(a, \theta_v^*)$), we get

$$kx = a(A_1\cos\theta_v^* + A_2\sin\theta_v^*),$$

where

$$\begin{aligned} A_1 &= k(\zeta)\{\exp(-\eta/2)\cos\delta\cdot\cos(n\zeta+\lambda) \\ &\qquad + \exp(\eta/2)\sin\delta\cdot\sin(n\zeta+\lambda)\}, \end{aligned}$$

$$\begin{aligned} A_2 &= k(\zeta)\{-\exp(-\eta/2)\cos\delta\cdot\sin(n\zeta+\lambda) \\ &\qquad + \exp(\eta/2)\sin\delta\cdot\cos(n\zeta+\lambda)\}, \\ A_1^2 + A_2^2 &= k^2[\exp(-\eta)\cdot\cos^2\delta \\ &\qquad + \exp(\eta)\cdot\sin^2\delta]. \end{aligned}$$

We introduced parameter $\lambda(\zeta)$ for the magnetic field line to be straight.

The QS conditions require: A_1 = const., A_2 = const. Using the condition $A_2(0) = 0$, the predictions of rather high Mercier stable $\langle\beta\rangle$ for Heliac-type QS systems with large number of periods ($N \geq 8$) were done in Ref.[12] which agrees with the later 3-D computations, Ref.[13]. Meanwhile, the 3-D computations of the ballooning modes have shown $\langle\beta\rangle_b$ to be smaller than Mercier limit both in Helias-like and Heliac-like QS stellarators in contradiction with some analytical necessary condition of the ballooning modes stability [14] (based on special probe perturbation [15]):

$$\frac{S^2}{2} + \left(\frac{\rho' R}{B_0^2 \iota^2}\right)\left[V_0''(\Phi)B_0^2\rho + \frac{(\iota_0\rho^3)'}{\iota\rho^2}\Delta\right] + \frac{1}{2}S\left(\frac{2p'R}{\iota^2 B_0^2}\right) > 0.$$

It follows from this criterion that ballooning modes slowly decreasing along the extended poloidal coordinate, are more stable than the Mercier modes, in systems with positive shear $S = a\iota'/\iota > 0$.

Such contradiction was discovered by Cooper, Hirshman and Lee [16]. They found unstable modes very localized in the extended poloidal representation, thus not localized strongly in radial direction. Later, the ballooning modes of such type were found to restrict the plasma pressure in a number of modern stellarator devices and future projects [17, 18], including the heliac-like quasi-helically symmetric stellarators, Ref.[19].

To understand a reason of the localized ballooning modes instability the role of magnetic surface geometry was investigated in Ref.[20] for conventional stellarator using the stellarator approximation. The 3D geometry of the magnetic surfaces and only 2D tokamak-like dipole secondary currents were taken into account. It was shown that only 3D geometrical effects are responsible for local ballooning modes instability. Up to now, there is no any analytical description of such kind of ballooning modes. Perhaps, the analysis of the terms in the local sufficient criterion, and the D. Lortz *et al.* criterion, can help to make the problem more clear.

To overcome the low ballooning stable β-limit it is to be refused from the condition of the toroidal quasi-symmetry by adding bumpiness into system. Accordingly a topology of the line B = const. was changed to poloidal one and rather high Mercier stable (4%) and ballooning stable (3%) betas were obtained in such Heliac-type quasi-mirror symmetrical system. For details see [19].

3. Normal Systems

Under "normal" we mean the toroidal systems in which the lines B = const. going along the all torus in toroidal or poloidal direction don't create the islands on a magnetic surface. In this case there is no local magnetic well, no locally trapped particles and corresponding enhanced transport. The formal condition of the "normality" could be formulated as a necessity of the straight lines B = const. in the flux coordinate with straight field lines [21]. In this case one can choose the coordinate surfaces θ = const. or ζ = const coinciding with surfaces B = const. Thus the condition of normality is as follows.

Toroidal direction:

$$\partial B/\partial\zeta = 0, \qquad B = B(a, \theta)$$

in any straight field line flux coordinates. In vector form:

$$\{(F+\partial\varphi/\partial\zeta)\boldsymbol{B}+\boldsymbol{B}\times\nabla\Psi)\}\cdot\nabla B=0.$$

Poloidal direction:

$$\partial B/\partial\theta=0; \qquad B=B(a,\zeta).$$

In vector form:

$$\{(J+\partial\varphi/\partial\theta)\boldsymbol{B}+\boldsymbol{B}\times\nabla\Psi)\}\cdot\nabla B=0.$$

These conditions differ from corresponding conditions of quasisymmetry by presence a free function φ, which obeys to the equation

$$\frac{\boldsymbol{B}\cdot\nabla\varphi}{2\pi}=B^2-\frac{<B^2>V'}{4\pi^2\sqrt{g}}.$$

Transforming coordinates θ, ζ to the habitual Boozer ones [5],

$$\theta=\theta_{\rm B}+\frac{\Psi'}{<B^2>V'}\varphi; \qquad \zeta=\zeta_B-\frac{\Phi'}{<B^2>V'}\varphi,$$

one could find

$$\frac{\partial B}{\partial\theta}=\frac{4\pi^2\sqrt{g}B^2}{<B^2>V'}\left\{\frac{\partial B}{\partial\theta_{\rm B}}-\frac{\Phi'}{<B^2>V'}\left(\frac{\partial B}{\partial\theta_{\rm B}}\frac{\partial\varphi}{\partial\zeta_{\rm B}}-\frac{\partial B}{\partial\zeta_{\rm B}}\frac{\partial\varphi}{\partial\theta_{\rm B}}\right)\right\}$$

$$\frac{\partial B}{\partial\zeta}=\frac{4\pi^2\sqrt{g}B^2}{<B^2>V'}\left\{\frac{\partial B}{\partial\zeta_{\rm B}}-\frac{\Psi'}{<B^2>V'}\left(\frac{\partial B}{\partial\theta_{\rm B}}\frac{\partial\varphi}{\partial\zeta_{\rm B}}-\frac{\partial B}{\partial\zeta_{\rm B}}\frac{\partial\varphi}{\partial\theta_{\rm B}}\right)\right\}$$

where $<B^2>V'=F\Phi'-J\Psi$.

An additional free function φ could provide fulfillment of the normality condition in the whole plasma volume. In near-axis approximation only one condition $A_1^2+A_2^2=$ const. should be fulfilled instead of two ones $A_1=$ const., $A_2=$ const. in the QS case.

Moreover, in contrast to the poloidal quasisymmetry, the poloidal normality can be satisfied at nonzero magnetic axis curvature, $k\neq 0$. The term $kx=k\rho\cos\theta$ in an expression for B near the magnetic axis could be canceled in the coordinate systems (a,θ^*,ζ^*) with the inclined coordinate surfaces $\zeta^*=$ const.:

$$\zeta=\zeta^*+a\gamma\cos\theta.$$

(In Boozer coordinates the linear in a term is impossible). In these coordinates

$$B=B_0(\zeta)(1+kaf\cos\theta)$$
$$=B_0(\zeta^*)+\left(\frac{\partial B_0}{\partial\zeta}\gamma+kfB_0\right)a\cos\theta.$$

The condition of the normality $B=B(a,\zeta^*)$ is fulfilled at

$$\frac{\partial B_0}{\partial\zeta}\gamma+kfB_0=0.$$

It follows from this condition that the magnetic axis curvature should be zero at the extremes of the B_0: $k=0$ at $\partial B_0/\partial\zeta=0$.

The poloidal normality seems to open interesting new possibilities for optimization of the helical magnetic systems.

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